FONDO ... ZOFALCONE

1881

BIBLIOTECA PROVINCIALE

Armadio XXVII

Palchetto 6

Num.° d'ordine 24

10775

19 e 56

NAZIONALE
B. Prov.
I
663
R. BIBLIOTECA VITT. EM. III
NAPOLI

Digitized by Google

P.P.
I
663

Digitized by Google

Digitized by Google

606829
SBN

METHODUS
INVENIENDI
LINEAS CURVAS
Maximi Minimive proprietate gaudentes,

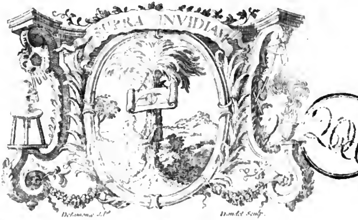
SIVE

SOLUTIO
PROBLEMATIS ISOPERIMETRICI
LATISSIMO SENSU ACCEPTI.

AUCTORE

LEONHARDO EULERO,

Profeſſore Regio, & Academiæ Imperialis Scientiarum PETROPOLITANÆ *Socio.*



LAUSANNÆ & GENEVÆ,

Apud MARCUM-MICHAELEM BOUSQUET & Socios.

MDCCXLIV.

Digitized by Google

Digitized by Google

METHODUS
INVENIENDI CURVAS
MAXIMI MINIMIVE PROPRIETATE
GAUDENTES.

CAPUT PRIMUM.

De Methodo maximorum & minimorum ad lineas curvas inveniendas applicata in genere.

DEFINITIO I.

1. M*ETHODUS maximorum & minimorum ad lineas curvas applicata*, eſt methodus inveniendi lineas curvas, quæ maximi minimive proprietate quapiam propoſita gaudeant.

COROLLARIUM I.

2. Reperiuntur igitur per hanc methodum lineæ curvæ, in quibus propoſita quæpiam quantitas maximum vel minimum obtineat valorem.

Digitized by Google

Coroll. II.

3. Cum autem eadem curva infinitis modis ſui ſimilis effici queat, Problema, niſi quædam reſtrictio adhibeatur, maxime eſſet indeterminatum, atque adeo nullum. Quæcunque enim curva præbeatur maximi minimive proprietate prædita, ſemper alia, illi quidem vel ſimilis vel diſſimilis, exhiberi poſſet, quæ illam proprietatem, vel majorem, vel minorem, in ſe contineret.

Coroll. III.

4. Quoniam igitur adæquata curvarum cognitio poſtulat, ut ea ad axem aliquem poſitione datum, ejuſque portiones quaſcunque quæ abſciſſæ vocantur, referantur: prima eaque præcipua reſtrictio ex quantitate abſciſſæ petenda erit.

Coroll. IV.

5. Problemata ergo ad methodum hanc pertinentia ita proponi debent, ut quærantur lineæ curvæ ad axem poſitione datum relatæ, quæ inter omnes alias curvas eidem abſciſſæ reſpondentes maximi minimive proprietate ſint præditæ.

SCHOLION.

6. Hæc itaque Methodus maximorum & minimorum maxime diſcrepat ab illa, quam alibi expoſuimus. Ibi enim, pro data ac determinata linea curva, locum determinavimus, ubi propoſita quædam quantitas variabilis ad curvam pertinens fiat maxima vel minima. Hic autem ipſa linea curva quæritur, in qua quantitas quædam propoſita fiat maxima vel minima. Methodus hæc jam ſuperiori Seculo, mox poſt inventam Analyſin infinitorum, excoli cœpit a Celeb. Fratribus Bernoulliis, atque ex eo tempore maxima cepit incrementa. Primum quidem Problema, quod ex hoc genere eſt tractatum, ad Mechanicam reſpiciebat, eoque quærebatur linea curva ſuper qua grave deſcendens citiſſime delabatur; cui *Curva brachyſtochrona* ſeu *Linea celerrimi deſcenſus* nomen erat impoſitum. In hoc Problemate jam

Digitized by Google

jam manifeſtum eſt, id, ſine adjuncta conditione, nequidem nomen quæſtionis retinere poſſe: perſpicuum enim eſt, quo brevior magiſque ad ſitum verticalem accedens linea capiatur, eo fore tempus deſcenſus ſuper ea brevius. Quamobrem non abſolute quæri poteſt linea, ſuper qua grave deſcendens celerrime ſeu breviſſimo tempore delabatur; ſed abſciſſæ quantitas, cui curva invenienda reſpondeat, ſimul debuit definiri; ita ut, inter omnes curvas eidem abſciſſæ in axe poſitione dato ſumtæ reſpondentes, quæreretur ea ſuper qua corpus grave citiſſime delaberetur. Neque vero in hoc Problemate iſta conditio ſufficiebat ad id determinatum efficiendum: ſed inſuper iſtam conditionem adjicere oportuit, ut curva invenienda per data duo puncta tranſeat; atque iſtud Problema his conditionibus adſtringi debuit, ut fieret determinatum, inter omnes, ſcilicet, lineas curvas per data duo puncta tranſeuntes eam determinare ſuper qua corpus deſcendens arcum datæ abſciſſæ reſpondentem breviſſimo tempore abſolvat. Interim tamen hic notandum eſt, conditionem tranſitus per duo puncta non eſſe abſolute neceſſariam, ſed in hoc Problemate per ipſam ſolutionem eſſe illatam. In ſolutione enim hujus Problematis immediate pervenitur ad æquationem differentialem ſecundi gradus, quæ bis integrata duas recipit conſtantes arbitrarias, ad quas determinandas duobus opus eſt punctis per quæ curva traducatur, vel aliis ſimilibus proprietatibus: atque hæc eadem conditio, quaſi ſua ſponte, ad omnia iſtiuſmodi Problemata accedit, quarum ſolutio immediate ad æquationem differentialem ſecundi gradus deducit. In Problematibus autem quæ reſolvuntur per æquationem differentialem quarti vel altioris ordinis, nequidem duo puncta ad curvam determinandam ſufficiunt, ſed tot opus eſt punctis, quot gradus differentialia obtinent. Contra vero, ſi ſolutio ſtatim ad æquationem algebraïcam perducat, tum ſine hujuſmodi conditione Problema perfecte erit determinatum; dummodo abſciſſæ longitudo definiatur. Verum hæc omnia clarius perſpicientur, quando infra ad ſolutiones Problematum perveniemus: ibique has notationes fuſius explicabimus. Hic enim in principio iſta tantum commemorare viſum eſt, ut

Digitized by Google

perverſas ideas circa determinationem hujuſmodi Problematum tollamus.

DEFINITIO II.

7. *Methodus maximorum ac minimorum abſoluta*, docet inter omnes omnino curvas, ad eandem abſciſſam relatas, determinare eam, in qua propoſita quædam quantitas variabilis maximum minimumve obtineat valorem.

COROLLARIUM.

8. In Problematibus igitur ad hanc methodum pertinentibus, datur axis poſitione; atque, inter omnes curvas quæ ad hunc axem ejuſque determinatam portionem referri poſſunt, determinatur ea in qua quantitas quædam variabilis ſit maxima vel minima.

SCHOLION.

9. Aliam conditionem ad maximi minimive determinationem, præter abſciſſæ quantitatem, hic in genere non adjicimus. Dantur enim Problemata, quæ hoc modo perfecte determinantur; quemadmodum infra diſtinctius patebit. Etſi enim etiam ejuſmodi Problemata occurrunt, ad quæ determinanda inſuper duo plurave puncta præſcribi poſſunt, per quæ quæſita curva tranſeat; tamen hoc demum ex ipſa cujuſcunque Problematis ſolutione perſpicietur. Namque ſi ad ejuſmodi æquationem pro curva quæſita perveniatur, in qua per integrationem novæ quantitates conſtantes ſint ingreſſæ, quæ in ipſa quæſtione non inerant; tum ſolutio cenſenda erit ambigua, atque vaga; eo quod innumerabiles lineas curvas, quæ ex determinatione illarum quantitatum conſtantium & arbitrariarum oriri poſſunt, in ſe complectitur. His igitur in caſibus erit concludendum, Problema ex ſua natura non penitus eſſe determinatum: ſed ad ejus plenam determinationem, præter abſciſſæ quantitatem, tot novas conditiones adjungi oportere, quibus illæ arbitrariæ conſtantes ad determinatos valores revocentur. Pro hujuſmodi autem conditionibus commodiſſime aſſumuntur puncta, per quæ curvæ quæſitæ

quæſitæ ſit tranſeundum; totidem vero puncta, quot inſunt in æquatione inventa quantitates arbitrariæ, ipſam æquationem determinatam reddent. Loco punctorum autem, ad curvam quæſitam perfecte determinandam, adhiberi etiam poſſunt totidem tangentes quæ curvam quæſitam tangant, &, ſi contactus debeat fieri in dato tangentis puncto, hæc conditio duobus punctis æquivalebit. Quin etiam in locum punctorum, aliæ quæcunque conditiones ſubſtitui poſſunt; dummodo eæ ita ſint comparatæ, ut per eas quantitates arbitrariæ in æquatione inventa contentæ determinentur. Neque vero ante opus eſt ſolutionem ad finem perducere, quam iſta dijudicatio ſuſcipiatur; ſed infra tradentur certa criteria, quorum ope, ſtatim, ex illa quantitate variabili quæ maximum minimumve eſſe debet, dignoſci poterit quæ novæ conſtantes in æquationem pro curva ingrediantur, quæ in quæſtione non continebantur. Oriuntur autem iſtæ conſtantes arbitrariæ ex gradu differentialium, ad quem æquatio pro curva quæſita exſurgit; quoti enim gradus prodit æquatio differentialis pro curva quæſita, tot quantitates arbitrariæ in illa cenſendæ ſunt poteſtate ineſſe; hincque totidem conditionibus opus erit ad curvam determinandam. Idem vero etiam uſu venit in ſolutione omnium Problematum, quando æquatio differentialis vel primi vel altioris gradus invenitur; ita ut hinc in præſenti inſtituto nulla peculiaris difficultas ineſſe cenſenda ſit.

DEFINITIO III.

10. *Methodus maximorum ac minimorum relativa* docet, non inter omnes omnino curvas eidem abſciſſæ reſpondentes, ſed inter eas tantum quæ præſcriptam quandam proprietatem communem habeant, eam determinare quæ maximi minimive proprietate gaudeat.

COROLL. I.

11. Ad hujuſmodi igitur Problemata ſolvenda, primum, ex omnibus omnino curvis eidem abſciſſæ reſpondentibus eæ ſunt

ſegregandæ, in quas eadem præſcripta proprietas competat; atque tum demum ex his ſegregatis ea quæ quæritur debebit definiri.

Coroll. II.

12. Quanquam autem tali conditione numerus curvarum omnium ad eandem abſciſſam relatarum vehementer reſtringitur; tamen is etiamnum manebit infinitus. Quin etiam, ſi non una, ſed plures proprietates præſcribantur, quibus omnes curvæ ex quibus quæſita eſt determinanda debeant eſſe præditæ; tamen uſque numerus curvarum manebit infinitus.

Coroll. III.

13. Quo plures itaque proponuntur proprietates, quæ iis curvis ex quibus quæſitam definiri oportet communes eſſe debeant; eo magis numerus curvarum inter quas electio quæſitæ eſt inſtituenda reſtringetur, etiamſi maneat infinitus.

SCHOLION I.

14. Ex hoc genere, in quo Methodum maximorum & minimorum relativam conſtituimus, initio hujus Seculi, primum a *Jacobo* BERNOULLIO in medium prolatum eſt famoſum illud *Problema Iſoperimetricum*; in quo quærebatur curva maximi minimive proprietate prædita, non inter omnes curvas ad eandem abſciſſam relatas, ſed inter eas tantum quæ ejuſdem eſſent longitudinis; ex quo iſtæ curvæ, ex quibus quæſitam erui opottebat, *iſoperimetræ* ſunt appellatæ. Ita ſi, inter omnes curvas eidem abſciſſæ reſpondentes & longitudine æquales, quæratur ea quæ cum abſciſſa & applicata maximum ſpatium includat; reperitur quæſito linea circularis ſatisfacere: quod quidem jam diu ante inventam hanc methodum Geometris innotuerat, ac demonſtratum erat. At, hoc caſu iterum, ex ipſa Problematum natura, novæ conditiones accedunt; uti in iis, quæ ad Methodum maximorum ac minimorum abſolutam pertinent;

quæ

Digitized by Google

quæ ex conſtantibus arbitrariis, quas ſolutio inducit, ſunt æſtimandæ. Ita in ſolutione Problematis, quo curva quæritur quæ inter omnes ejuſdem longitudinis maximam comprehendat aream cum abſciſſa, duæ conſtantes novæ ingrediuntur; ex quo, ad Problema determinatum efficiendum, id ita eſt proponendum, ut inter omnes curvas ejuſdem longitudinis, quæ non ſolum eidem abſciſſæ reſpondeant, ſed etiam per data duo puncta tranſeant, quæratur ea quæ ad datam abſciſſam maximam aream referat. Atque ſimili modo evenire poteſt, ut quatuor puncta, & plura etiam interdum, pro arbitrio aſſumi debeant, quo Problema fiat determinatum: cujus rei dijudicatio ex ipſa Problematum natura eſt petenda. Quemadmodum autem, in Problemate iſoperimetrico, omnes curvæ ex quibus quæſitam determinari oportet ejuſdem longitudinis ponuntur; ita loco hujus proprietatis alia quæcunque proponi poteſt, quæ omnibus communis eſſe debeat. Sic jam quæſitæ ſunt curvæ maximi minimive proprietate prædita, inter omnes eas curvas ad eandem abſciſſam relatas tantum, quæ circa eam abſciſſam converſæ omnes æquales ſuperficies generent; atque ſimili modo aliæ quæcunque proprietates proponi poſſunt. Deinde etiam, non una, ſed plures hujuſmodi proprietates præſcribi poſſunt, quæ omnibus curvis inter quas ea quæ maximum minimumve aliquod contineat definienda ſit communes eſſe debeant. Ita ſi quæreretur curva maximi vel minimi proprietate quapiam prædita, inter omnes curvas eidem abſciſſæ reſpondentes, quæ tam eſſent omnes inter ſe longitudine æquales, quam etiam areas æquales concluderent.

S C H O L I O N II.

15. Propter hoc diſcrimen inter Methodum maximorum & minimorum abſolutam ac relativam, tractatio noſtra erit bipartita. Primum ſcilicet methodum trademus, inter omnes omnino curvas eidem abſciſſæ reſpondentes, eam determinandi quæ maximi minimive proprietate ſit prædita. Deinde vero progrediemur ad ejuſmodi Problemata, in quibus curva maximi minimive proprietate

prietate gaudens poſtulatur, inter omnes curvas quæ unam pluresve propoſitas proprietates communes habeant; atque ex numero harum proprietatum iſtius tractationis denuo ſubdiviſio orietur. Interim tamen non opus erit in hac ſubdiviſione longius progredi; cum mox reperiatur methodus, quotcunque etiam propoſitæ fuerint proprietates, Problemata facile reſolvendi. Solutiones enim Problematum prima fronte maxime intricatorum præter opinionem fient perquam expeditæ, ac levi calculo abſolvendæ.

HYPOTHESIS I.

16. *In hac tractatione abſciſſam, ad quam omnes curvas referemus, perpetuo littera* x, *applicatam vero littera* y *deſignabimus. Tum vero, ſumptis elementis abſciſſæ æqualibus, ſemper erit* $dy = pdx$; $dp = qdx$; $dq = rdx$; $dr = sdx$; *&c.*

COROLL. I.

17. His igitur ſubſtitutionibus omnia differentialia ipſius *y* cujuſcunque gradus ex expreſſionibus tollentur, atque præter differentiale dx nulla alia differentialia relinquentur. Quanquam autem hoc modo omnia differentialia, præter dx, ſpecie tantum, non revera tolluntur; tamen hæ ſubſtitutiones ingens nobis in præſenti inſtituto afferent ſubſidium.

COROLL. II.

18. Quin etiam hujuſmodi ſubſtitutionibus differentialis conſtantis aſſumtio penitus de calculo tollitur: quodcunque enim differentiale aliud conſtans aſſumatur, poſt iſtas ſubſtitutiones perpetuo eadem formula emergere debet. Interim tamen, ob methodum infra adhibendam, neceſſe erit differentiale dx tanquam conſtans aſſumere.

COROL.

Coroll. III.

19. Ut autem facilius appareat, quomodo per has ſubſtitutiones differentialia cujuſque gradus ipſius y evaneſcant; juvabit ſequentem Tabellam adjeciſſe.

$$\begin{aligned} dy &= p\,dx \\ ddy &= dp\,dx = q\,dx^2 \\ d^3y &= dq\,dx^2 = r\,dx^3 \\ d^4y &= dr\,dx^3 = s\,dx^4 \\ d^5y &= ds\,dx^4 = t\,dx^5 \\ &\&c. \quad \&c. \quad \&c. \end{aligned}$$

Coroll. IV.

20. Quod ſi etiam arcus curvæ abſciſſæ x reſpondens, cum ſuis differentialibus cujuſcunque gradus occurrat; ea omnia per iſtas litteras ita exprimi poterunt, ut nulla alia differentialia præter dx adſint. Poſito enim arcu $= w$ erit.

$$\begin{aligned} w &= \int\sqrt{(dx^2+dy^2)} = \int dx\sqrt{(1+pp)} \\ dw &= dx\sqrt{(1+pp)} \\ ddw &= \frac{pq\,dx^2}{\sqrt{(1+pp)}} \\ d^3w &= \frac{pr\,dx^3}{\sqrt{(1+pp)}} + \frac{qq\,dx^3}{(1+pp)^{3:2}} \\ &\&c. \end{aligned}$$

Coroll. V.

21. Simili modo, ex his radius oſculi ſeu curvedinis curvæ, in quovis loco, per quantitates ſpecie ſaltem finitas poterit exprimi. Cum enim, poſito elemento dx conſtante, ſit longitudo radii oſculi $= \frac{-dw^3}{dx\,ddy}$; fiet ea $= \frac{-(1+pp)^{3:2}}{q}$

Digitized by Google

COROLL. VI.

22. Porro ex iiſdem ſubſtitutionibus erit, ut ſequitur.

$$\text{Subtangens} = \frac{y\,dx}{dy} = \frac{y}{p}$$

$$\text{Subnormalis} = \frac{y\,dy}{dx} = py$$

$$\text{Tangens} = \frac{y\,dw}{dy} = \frac{y\surd(1+pp)}{p}$$

$$\text{Normalis} = \frac{y\,dw}{dx} = y\surd(1+pp)$$

Atque, pari modo, omnes quantitates finitæ ad curvam pertinentes, niſi integralia involvant, per hujuſmodi quantitates finitas ita exprimi poterunt, ut nulla differentialia amplius ineſſe videantur.

DEFINITIO IV.

23. *Maximi minimive Formula*, pro quovis Problemate, nobis erit ea quantitas, quæ in curva quæſita maximum minimumve valorem obtinere debet.

COROLL. I.

24. Quoniam in omnibus Problematibus ad quæ hæc Methodus eſt accommodata, curva quæritur quæ, vel inter omnes, vel tantum inter innumeras curvas certo modo determinatas, maximi minimive proprietate gaudeat; hæc ipſa proprietas, quæ in curva quæſita maxima vel minima eſſe debet, erit quantitas; eaque exprimetur Formula, quam maximi minimive Formulam hic appellamus.

C o-

Digitized by Google

Coroll. II.

25. Cum autem maximi minimive proprietas ita proponi debeat, ut ad datam ac determinatam abſciſſam referatur; Formula maximi minimive quoque ad illam definitam abſciſſam debet referri.

Coroll. III.

26. Erit igitur maximi minimive Formula, quantitas variabilis a longitudine abſciſſæ cujuſcunque cui reſpondet pendens. Atque in quovis Problemate quæretur curva, pro qua, ad definitam abſciſſam, illa maximi minimive Formula maximum minimumve obtineat valorem.

Coroll. IV.

27. Neque vero maximi minimive Formula a ſola abſciſſa pendere poteſt: hoc enim ſi eſſet, pro omnibus curvis eidem abſciſſæ reſpondentibus eundem obtineret valorem, atque idcirco omnes æqualiter ſatisfacerent.

Coroll. V.

28. Hanc ob rem quoque maximi minimive Formula, præter abſciſſam omnibus curvis quæ in conſiderationem veniunt communem, a qualibet curva peculiariter debet pendere; ita ut una ſit, pro qua maximum minimumve valorem induere queat.

SCHOLION I.

29. Quo hæc omnia clarius intelligantur, atque ſtatus Quæſtionum in ſequenti pertractandarum melius comprehendatur; ponamus, vel inter omnes omnino curvas, vel tantum inter innumerabiles certam quamdam proprietatem communem habentes, quæ eidem abſciſſæ A Z reſpondeant, eam determinari debere, *Fig. 1.*

Digitized by Google

Fig. 1. pro qua valor formulæ W ſit maximus vel minimus. Ponamus huic Quæſtioni ſatisfacere curvam a m z, ita ut, quæcunque alia curva ad abſciſſam definitam A Z referatur, valor formulæ W vel fiat minor quam pro hac curva, vel major: prout in curva ſatisfaciente, W vel maximum eſſe debet vel minimum. In hac igitur quæſtione latiſſime patente, habemus primo abſciſſam determinatæ longitudinis A Z: deinde curva eſt quærenda, vel inter omnes omnino curvas ad eandem hanc abſciſſam relatas, vel tantum inter innumerabiles quibus una pluresve proprietates ſint communes, prout quæſtio ad methodum maximorum & minimorum vel abſolutam vel relativam eſt accommodata: tertiò habemus eam quantitatem W, cujus valor in curva quæſita a m z maximus eſſe debet vel minimus; eritque igitur quantitas W maximi minimive formula, ſicut ea eſt definita. Nunc igitur ſtatim apparet hanc formulam W ita eſſe debere comparatam, ut ad omnes curvas quæ quidem concipi poſſunt accommodari queat. Primo ſcilicet a quantitate abſciſſæ definitæ A Z debebit pendere, ita ut ea mutetur, valore ipſius A Z mutato. Deinde etiam a natura cujuſvis curvæ quæ quidem concipi poteſt peculiari modo debet pendere: niſi enim ita eſſet comparata, pro omnibus curvis eundem valorem ſortiretur, quæſtioque foret nulla. Quamobrem quantitas W, præter abſciſſam, in ſe quoque complecti debebit quantitates ad curvam ipſam pertinentes. Cum igitur omnis curva determinetur per relationem inter abſciſſam & applicatam, quantitas W debet eſſe conflata ex abſciſſa & applicata, & quantitatibus inde pendentibus. Hoc eſt, ſi abſciſſa indefinita ponatur $= x$, & applicata reſpondens indefinita $= y$; quantitas W eſſe debet functio binarum variabilium x & y. Quod cum ita ſit, ſi curva quæcunque determinata concipiatur, atque ex ejus natura relatio inter y & x in formula W ſubſtituatur, ea definitum impetrabit valorem ad datam illam curvam atque ejus definitam abſciſſam pertinentem. Quoniam jam, pro aliis atque aliis curvis, formula W diverſos valores induit, etiamſi in omnibus abſciſſa eadem capiatur; manifeſtum eſt inter innumerabiles illas curvas unam.

unam eſſe debere in qua valor formulæ *W* maximus fiat vel minimus; atque ad hanc curvam pro data quacunque determinata quæſtione inveniendam, Methodus tradenda eſt comparata.

Coroll. VI.

30. Erit igitur maximi minimive formula *W*, functio quædam binarum variabilium x & y: quarum altera x abſciſſam, altera y applicatam denotat. In *W* ineſſe igitur poterunt, non ſolum ipſæ variabiles x & y, ſed etiam omnes quantitates ab iis pendentes, cujuſmodi ſunt p, q, r, s, &c. quarum ſignificationes ſupra tradidimus. Quinetiam formulæ integrales ex his ortæ quæcunque in *W* ineſſe poſſunt: imo etiam debent; ſiquidem quæſtio debeat eſſe determinata, uti mox oſtendemus.

Coroll. VII.

31. Propoſita igitur ejuſmodi formula *W*, ſeu functione ipſarum x & y, ſi quæſtio ad methodum maximorum & minimorum abſolutam pertineat, ejuſmodi æquatio inter x & y deſideratur, ut, ſi in *W* valor ipſius y per x determinatus ſubſtituatur, atque ipſi x valor definitus tribuatur; major prodeat quantitas pro *W*, vel minor, quam ſi ulla alia æquatio inter x & y aſſumta fuiſſet.

Coroll. VIII.

32. Hoc ergo pacto, quæſtiones ad doctrinam linearum curvarum pertinentes ad Analyſin puram revocari poſſunt. Atque viciſſim, ſi hujus generis quæſtio in Analyſi pura ſit propoſita, ea ad doctrinam de lineis curvis poterit referri ac reſolvi.

SCHOLION II.

33. Quanquam hujus generis quæſtiones ad puram Analyſin

 redu-

reduci poſſunt, tamen expedit eas cum doctrina linearum curvarum conjungere. Quod ſi enim animum a lineis curvis abducere, atque ad ſolas quantitates abſolutas firmare velimus; quæſtiones primum ipſæ admodum fierent abſtruſæ & inelegantes, uſuſque earum ac dignitas minus conſpiceretur: Deinde etiam methodus reſolvendi hujuſmodi quæſtiones, ſi in ſolis quantitatibus abſtractis proponeretur, nimium foret abſtruſa & moleſta; cum tamen eadem, per inſpectionem figurarum & quantitatum repræſentationem linearem, mirifice adjuvetur atque intellectu facilis reddatur. Hanc ob cauſam, etſi hujus generis quæſtiones, cum ad quantitates abſtractas, tum concretas applicari poſſunt, tamen eas ad lineas curvas commodiſſime traducemus & reſolvemus. Scilicet quoties æquatio ejuſmodi inter x & y quæritur, ut formula quædam propoſita & compoſita ex x & y, ſi ex illa æquatione quæſita valor ipſius y ſubrogetur, & ipſi x determinatus valor tribuatur, maxima fiat vel minima: tum ſemper quæſtionem transferemus ad inventionem lineæ curvæ, cujus abſciſſa ſit x, & applicata y, pro qua illa formula W fiat maxima vel minima, ſi abſciſſa x datæ magnitudinis capiatur. His igitur notatis, natura hujuſmodi quæſtionum ſatis luculenter perſpicitur: niſi forte cuiquam adhuc dubium creat ambigua locutio de maximo & minimo ſimul. Verum ne hic quidem ulla adeſt ambiguitas; nam etſi methodus ipſa æque monſtrat maxima & minima, tamen in quovis caſu facile erit diſcernere, utrum ſolutio præbeat maximum an minimum. Sæpe numero autem evenire poteſt, ut in data quæſtione tam maximum quam minimum locum obtineat, atque his caſibus ſolutio erit duplex, altera monſtrante maximum, altera minimum. Plerumque autem alterutrum, ſcilicet vel maximum vel minimum ſolet eſſe impoſſibile; quod evenit, ſi maximi minimive formula in infinitum vel creſcere vel decreſcere poteſt; his enim caſibus, vel non dabitur maximum, vel non minimum. Uſu venire etiam poteſt, ut formula propoſita W in infinitum tam creſcere quam decreſcere queat, atque his caſibus nulla prorſus ſolutio locum habe-

habebit. Hæc autem diſcrimina cuncta ipſe calculus poſt ſolutionem perpetuo monſtrabit.

PROPOSITIO I. THEOREMA.

34. *Ut per maximi minimive formulam* W, *curva determinetur* amz, *quæ præ omnibus reliquis ſatisfaciat, formula* W *debet eſſe quantitas integralis indefinita, quæ, niſi data aſſumatur relatio inter* x *&* y, *integrari nequeat.*

DEMONSTRATIO.

Ponamus enim formulam W integralia indefinita non involvere; erit ea functio quantitatum x & y, indeque pendentium p, q, r, s, &c. vel algebraïca, vel talis tranſcendens quæ ſine aſſumta relatione inter x & y exhiberi poſſit; quod evenit, ſi vel logarithmi harum quantitatum, vel arcus circulares, vel aliæ hujuſmodi quantitates tranſcendentes definitæ ingrediantur, quæ algebraïcis æquivalentes ſunt cenſendæ. Quod ſi jam W ponatur functio talis ipſarum x & y tantum, manifeſtum eſt valorem formulæ W, quem pro data curva amz ad datam abſciſſam AZ relata obtinet, tantum ab ultima applicata Zz pendere; atque pro omnibus curvis in Z eandem applicatam Zz habentibus fore eundem; atque adeo tali formula W indoles totius curvæ non determinabitur, ſed tantum poſitio extremi ejus puncti z; ſi in W præter x & y etiam quantitas p inſit, tum præter longitudinem applicatæ Zz poſitio tangentis curvæ in z, ſeu poſitio ultimi elementi in z determinabitur. Sin autem inſuper q ingrediatur, tum poſitio binorum elementorum curvæ contiguorum in z determinabitur, & ita porro. Ex quibus ſequitur, ſi fuerit W functio determinata ipſarum x, y, p, q, r, &c. tum per illam tantum curvæ portionem infinite parvam circa extremitatem z determinari: atque pro omnibus curvis in eandem extremitatem deſinentibus eundem valorem ipſius W eſſe proditurum. Ut itaque per formulam W tota curva amz, quatenus toti abſciſſæ AZ reſpondet, definiatur, formulam

mulam W ita oportet eſſe comparatam, ut ejus valor ad determinatam curvam a m z applicatus, a poſitione ſingulorum elementorum hujus curvæ intra terminos a & z ſitorum pendeat. Hoc autem evenire non poteſt, niſi quantitas W ſit formula integralis indefinita, quæ generatim ſine aſſumta æquatione inter x & y integrationem non admittat. *Q. E. D.*

COROLL. I.

35. Niſi igitur maximi minimive formula W ſit quantitas integralis indefinita, nequidem linea curva in qua valor ipſius W ſit maximus vel minimus determinabitur; atque adeo quæſtio de invenienda curva, in qua eſſet W maximum vel minimum erit nulla.

COROLL. II.

36. Ut igitur curva aſſignari poſſit, in qua valor ipſius W præ aliis ſit maximus vel minimus, formula W talem formam $\int Z dx$ habere debet; atque quantitatem Z ita comparatam eſſe oportet ut differentiale $Z dx$, niſi æquatio ſtatuatur inter x & y, integrari nequeat.

SCHOLION.

37. Quoniam maximi minimive formula W debet eſſe integrale formulæ differentialis indefinitæ primi gradus; hoc eſt cujus integrale fiat quantitas finita; ea formula differentialis ſemper ad hujuſmodi formam $Z dx$ poterit reduci, ope litterarum p, q, r, &c. Et hanc ob rem in ſequentibus maximi minimive formula perpetuo per $\int Z dx$ nobis indicabitur. Erit autem Z functio non ſolum quantitatum x & y, ſed etiam continebit litteras p, q, r, &c. Ita ſi area A a z Z debeat eſſe maxima vel minima, formula W abibit in $\int y dx$; &, ſi ſuperficies ſolidi rotundi quod generatur rotatione curvæ a m z circa axem A Z debeat eſſe maxima vel minima, erit $W = \int y dx \sqrt{(1 + pp)}$;

atque

atque ita porro quæcunque formula debeat in curva quæſita eſſe maxima vel minima, ea ſemper erit hujus formæ $\int Z\,dx$, ſcilicet integrale quantitatis finitæ cujuſdam Z in differentiale dx ductæ. Debet autem Z ejuſmodi eſſe quantitas, ut ſi æquatio ſtatuatur inter x & y, integrale $\int Z\,dx$ determinatum obtineat valorem: ex quo Z erit functio quantitatum x, y, & inde pendentium p, q, r, &c. vel algebraïca ſive determinata, vel prætérea ipſa in ſe complectetur formulas integrales indeterminatas; quod diſcrimen probe eſt tenendum. Ita ſi maximi minimive formula W fuerit $\int y\,dx$, vel $\int y\,dx\sqrt{(1+pp)}$; quantitas Z erit algebraïca, at ſi ſit $W = \int yx\,dx \int y\,dx$, tum erit $Z = yx\int y dx$, hoc eſt ipſa quantitas Z erit indeterminata, cujus valor niſi relatio inter x & y detur, exhiberi nequit. Quin etiam evenire poteſt, ut valor ipſius Z hujuſmodi formula evoluta exprimi nequeat, ſed tantum per æquationem differentialem demum erui debeat, ut ſi fuerit $dZ = y\,dx + ZZ\,dx$; ex qua æquatione valor ipſius Z per x & y nequidem exhiberi poteſt. Hinc igitur tria naſcuntur genera formularum $\int Z\,dx$, quæ in curvis quæſitis maxima vel minima fieri debent. Quorum primum eas complectitur formulas, in quibus Z eſt functio algebraïca ſeu determinata ipſarum x, y, & p, q, r, &c. Ad ſecundum genus referimus eas formulas, in quibus quantitas Z ipſa inſuper formulas integrales involvit. In tertio autem genere continentur eæ formulæ, in quibus valor ipſius Z per æquationem differentialem cujus integratio non conſtat determinatur.

PROPOSITIO II. THEOREMA.

38. *Si fuerit* amz *curva, in qua valor formulæ* $\int Z\,dx$ *ſit maximus vel minimus, atque* Z *ſit functio algebraïca ſeu determinata ipſarum* x, y, p, q, r, &c. *tum ejuſdem curvæ quæcunque portio* mn *eadem gaudebit prærogativa, ut pro ea ad ſuam abſciſſam* MN *relata, valor ipſius* $\int Z\,dx$ *ſit pariter maximus vel minimus.*

DEMONSTRATIO.

Valor formulæ $\int Z dx$ pro abſciſſa A Z eſt aggregatum omnium valorum ejuſdem formulæ, qui ſingulis abſciſſæ A Z portionibus reſpondent. Quod ſi ergo abſciſſa A Z in partes quotcunque, quarum una ſit M N, diviſa concipiatur, atque ad ſingulas partes haſce valor formulæ $\int Z dx$ exhibeatur; ſumma omnium horum valorum præbebit valorem formulæ $\int Z dx$, qui toti abſciſſæ A Z convenit; & qui erit maximus vel minimus. Quoniam autem Z ponitur functio algebraïca ipſarum x, y, p, q, &c. valor formulæ $\int Z dx$ reſpondens abſciſſæ portioni M N, a ſola portionis curvæ reſpondentis m n indole pendebit, idemque manebit, utcunque reliquæ partes a m & n z varientur; ſingularum enim litterarum x, y, p, q, &c. valores per ſolam curvæ portionem m n determinantur. Si ergo formulæ $\int Z dx$ valores, qui conveniunt abſciſſæ portionibus A M, M N, N Z, ponantur P, Q & R, quantitates hæ P, Q, & R, a ſe mutuo non pendebunt. Quare cum earum aggregatum $P + Q + R$ ſit maximum vel minimum, etiam unaquaque maximi minimive proprietate prædita ſit neceſſe eſt. Hanc ob rem, ſi in curva a m z formula $\int Z dx$ maximum minimumve habeat valorem, & quantitas Z ſit functio algebraïca ipſarum x, y, p, q, &c. tum etiam, pro qualibet illius curvæ portione, eadem formula $\int Z dx$ maximi minimive proprietate gaudebit. *Q. E. D.*

COROLL. I.

39. Quod ſi ergo curva fuerit inventa a m z, quæ pro abſciſſa data A Z, habeat valorem formulæ $\int Z dx$ maximum vel minimum, atque Z ſit functio algebraïca ſeu determinata, tum etiam ejuſdem curvæ quælibet portio, reſpectu abſciſſæ ſuæ reſpondentis, eadem maximi minimive proprietate gaudebit.

Digitized by Google

C o r o l l. II.

40. In hujuſmodi igitur Problematibus, ubi tale maximum minimumve quæritur, non opus eſt quantitatem abſciſſæ, cui maximum minimumve reſpondeat, definire; ſed ſi, pro una quacunque abſciſſa, formula $\int Zdx$ ſit maximum vel minimum, tum eadem pro quacunque alia abſciſſa eadem proprietate gaudebit.

C o r o l l. III.

41. Hujuſmodi igitur Problemata reſolventur, ſi ſingulæ curvæ quæſitæ particulæ ita determinentur, ut pro iis valor formulæ $\int Zdx$ fiat maximus vel minimus. Tum enim ſimul tota curva, & quæcunque ejus portio, pariter eadem maximi minimive proprietate erit inſtructa.

S C H O L I O N.

42. Proprietas hæc, qua gaudent curvæ in quibus iſtius modi formulæ $\int Zdx$, ubi Z eſt functio algebraïca ſeu determinata ipſarum x, y, p, q, &c. ſunt maximum vel minimum, eſt maximi momenti; ea enim innititur univerſa methodus hujus generis Problemata reſolvendi. Ideo autem potiſſimum hanc Propoſitionem afferre viſum eſt, ne ea proprietas, quæ his tantum formulis $\int Zdx$, ubi Z eſt functio vel algebraïca vel determinata, eſt propria, omnium omnino formularum quæ proponi poſſunt communis eſſe putetur: in ſequente enim Propoſitione demonſtrabimus, ſi in Z inſint formulæ integrales; tum eandem proprietatem non amplius locum habere: ex quo ſimul natura hujuſmodi quæſtionum clarius intelligetur. Hujus autem præſentis Propoſitionis demonſtratio ex eo petita eſt fundamento, quod valor formulæ $\int Zdx$, ſiquidem Z eſt functio vel algebraïca vel determinata ipſarum x, y, p, q, r, &c. qui convenit cuicunque abſciſſæ portioni MN, a ſola curvæ portione reſpondente mn pendeat, neque a reliqua curva, vel anteriore am, vel poſteriore nz afficiatur: quæ ratio ceſſat, ſi in Z inſint for-

mulæ integrales indeterminatæ. Valores enim quantitatum x, y, p, q, r, &c. qui pro arcu curvæ m n obtinent, tantum a poſitione elementorum hujus arcus m n, atque elementis aliquot contiguis quæ arcum finitæ quantitatis non conſtituunt pendent; ex quo etiam quantitas ex iis litteris utcunque compoſita per ſolam arcus m n indolem determinabitur, niſi adfuerint quantitates integrales, cujuſmodi ſunt $\int y\,dx$, quæ totam aream anteriorem A a m M introduceret, vel $\int dx\sqrt{(1+pp)}$, quæ totum arcum præcedentem a m involveret. Hinc igitur diſtinctius intelligitur, quid per functionem determinatam ipſarum x, y, p, q, r, &c. denotare velimus: Functio ſcilicet determinata ita eſt comparata, ut, pro quovis loco, a præſentibus valoribus litterarum x, y, p, q, &c. tantum pendeat, neque valores earum anteriores in ſe complectatur. Functio autem indeterminata eſt talis, cujus valor in quovis loco, non ex ſolis valoribus quos hæ litteræ x, y, p, q, &c. in iſto loco obtinent determinari poteſt, ſed inſuper omnes valores ad ſui determinationem requirit, quos iſtæ litteræ in omnibus locis anterioribus obtinuerunt. Ita patet, omnes functiones algebraïcas eſſe ſimul determinatas; præterea vero etiam omnes functiones tranſcendentes, quæ a relatione inter x & y non pendent ſunt determinatæ, cujuſmodi ſunt, $l\sqrt{(xx+yy)}$, e^{py}, Aſin. $\frac{py}{q}$; quarum valores in quovis loco ex valoribus litterarum, quos in hoc ſolo loco obtinent, aſſignari poſſunt. Quando autem in functione quapiam inſunt formulæ integrales indeterminatæ, quæ a mutua relatione inter x & y, quam ubique tenent, pendent, tum earum valor, in dato loco, non ex valoribus, quos hæ litteræ in iſto loco habent, cognoſci poteſt, ſed inſuper omnes valores in locis quibuſque anterioribus noſſe oportet, hoc eſt generalem relationem inter coordinatas x & y: taleſque functiones vocamus indeterminatas; quippe quæ toto cœlo diverſæ ſunt ab iis, quas determinatas appellavimus.

PRO-

Digitized by Google

Propositio III. Theorema.

43. *Si fuerit* amz *curva abſciſſa* AZ *reſpondens, in qua* $\int Z dx$ *ſit maximum vel minimum, in* Z *autem contineantur formulæ integrales indeterminatæ; tum eadem maximi minimive proprietas non cadit in quamlibet curvæ portionem, ſed toti tantum curvæ abſciſſæ* AZ *reſpondenti propria erit.*

Demonſtratio.

Concipiatur tota curva amz, pro qua $\int Z dx$ eſt maximum vel minimum, in duas partes quaſque diviſa per applicatam Mm; ſitque formulæ $\int Z dx$ valor conveniens portioni am $= P$, ejuſdem autem formulæ valor pro altera portione mz ſit $= Q$: pro tota igitur curva amz valor formulæ $\int Z dx$ erit $= P + Q$, quem ponimus eſſe maximum vel minimum. Quo autem omnem ambiguitatem tollamus, totamque rem diſtinctius proponere queamus; ponamus $P + Q$ eſſe maximum: quod enim de maximo demonſtrabitur, idem de minimo facile intelligetur. Quod ſi jam valor ipſius Q a valore ipſius P non penderet, tum aggregatum $P + Q$ maximum eſſe non poſſet, niſi ſimul uterque valor P & Q ſeorſim ſit maximus. At noſtro caſu, quo quantitas Z in ſe continet formulas integrales indeterminatas, valor ipſius Q non tantum a curvæ portione mz ad quam refertur pendebit, ſed ſimul a tota curva anteriore am; atque adeo a valore ipſius P. Nunc dicimus, ad id ut $P + Q$ ſit maximum, non requiri, ut valor ipſius P ſit maximus. Ponamus enim portionem curvæ am ita eſſe comparatam, ut pro ea P ſit maximum, & aliquantillum mutari concipiatur portio curvæ am, ita ut valor formulæ $\int Z dx$ minor evadat, puta $= P - p$: fieri utique poterit ut ex hac mutatione valor ipſius Q creſcat, quod incrementum ponatur q: eritque, mutata aliquantillum portione am, ita ut pro ea $\int Z dx$ non amplius ſit maximum, valor formulæ $\int Z dx$ pro tota curva amz $= P - p + Q + q$. Cum igitur evenire queat ut ſit $q > p$, intelligitur formu-

lam $\int Zdx$ pro tota curva a m z maximam eſſe poſſe, etiamſi maxima non ſit pro qualibet portione a m. *Q. E. D.*

COROLL. I.

44. Quando ergo curva fuerit inventa, quæ, pro data abſciſſa A Z, habeat valorem formulæ $\int Zdx$ maximum vel minimum, & Z ſit functio indeterminata; tum non ſequitur quamlibet curvæ inventæ portionem eadem maximi minimive proprietate fore præditam.

COROLL. II.

45. In reſolutione igitur hujuſmodi Problematum, in quibus curva quæritur, quæ pro data abſciſſa A Z habeat $\int Zdx$ maximum vel minimum, perpetuo ad totius abſciſſæ propoſitæ quantitatem erit reſpiciendum, atque maximum vel minimum ad eam tantum, non vero ad ejus quamlibet portionem, accommodari debebit.

COROLL. III.

46. Maximum igitur hinc patet diſcrimen, quod inter formulas $\int Zdx$, in quibus Z functio eſt determinata vel indeterminata, intercedit; ſimulque autem Methodorum diverſitas intelligitur, quibus ad reſolutiones quæſtionum, in quibus hujuſmodi formularum maximi minimive valores requiruntur, uti oportebit.

SCHOLION.

47. Ex demonſtratione hujus Propoſitionis non quidem neceſſario ſequitur, ſi pro data abſciſſa A Z curva habeat formulam $\int Zdx$ maximam vel minimam, tum ſingulas ejus portiones eadem hac prærogativa gaudere; verumtamen ſatis intelligitur, quoties eadem proprietas in ſingulas portiones competat, id ca-

ſu

Digitized by Google

ſu evenire. Hincque nihilominus ſumme neceſſarium eſt, ſolutionem perpetuo ad totam propoſitam abſciſſam accommodare. Interim tamen, in Problematibus ad methodum relativam pertinentibus evenire poteſt, ut formulas $\int Z dx$, in quibus Z ſit functio indeterminata, quaſi determinata eſſet tractare liceat. Hoc ſcilicet accidit, ſi inter omnes tantum curvas in quibus formulæ illæ integrales indeterminatæ quæ in Z inſunt æquales obtinent valores, ea deſideretur, in qua $\int Z dx$ ſit maximum vel minimum : hoc enim caſu formulæ illæ integrales indeterminatæ fieri cenſendæ ſunt determinatæ. Ita ſi, inter omnes curvas ejuſdem longitudinis, determinanda ſit ea in qua ſit $\int Z dx$ maximum vel minimum, atque in Z præter quantitates determinatas, inſit arcus curvæ $\int dx \surd (1 + pp)$; hic, quia in omnibus curvis ex quibus quæſitam definire oportet, eundem obtinet valorem, inſtar functionis determinatæ tractari poterit: Hæc autem cuncta in ſequentibus clarius explicabuntur.

HYPOTHESIS II.

48. *Si curva abſciſſa* AZ *in elementa innumerabilia infinite parva & inter ſe æqualia diſſecetur, cujuſmodi ſunt* IK, KL, LM, *&c. atque portio quæcunque* AM *vocetur* x, *cui reſpondeat functio quæcunque variabilis* F, *eandem functionem* F, *quatenus referetur ad puncta abſciſſæ vel ſequentia* N, O, P, Q, *&c. vel antecedentia* L, K, I, *&c. ita denotabimus, ut ſit valor iſtius functionis qui pro puncto* M *eſt* = F, *ut ſequitur.* Fig. 2.

$$\left.\begin{array}{l} \textit{pro}\ N = F' \\ \textit{pro}\ O = F'' \\ \textit{pro}\ P = F''' \\ \textit{pro}\ Q = F^{IV} \\ \textit{pro}\ R = F^{V} \end{array}\right\}\ \textit{pro punctis abſciſſæ ſequentibus}$$

&c.

pro

$$\left.\begin{array}{l} pro\ L = F_{,} \\ pro\ K = F_{,,} \\ pro\ I = F_{,,,} \\ pro\ H = F_{IV} \\ \quad \&c. \end{array}\right\} \textit{pro punctis abſciſſæ antecedentibus.}$$

Atque hoc pacto, ſine prolixa differentialium ſcriptione, valor functionis cujuſcunque variabilis, qui in quovis abſciſſæ puncto locum obtinet, commode indicabitur.

C o r o l l. I.

49. Cum igitur functionis cujuſque valor, in loco quocunque, ſit æqualis ſuo valori in loco antecedente differentiali ſuo aucto, erit

$$\begin{array}{l|l} F' = F + dF & F = F_{,} + dF_{,} \\ F'' = F' + dF' & F_{,} = F_{,,} + dF_{,,} \\ F''' = F'' + dF'' & F_{,,} = F_{,,,} + dF_{,,,} \\ F^{IV} = F''' + dF''' & F_{,,,} = F_{IV} + dF_{IV} \\ \quad \&c. & \quad \&c. \end{array}$$

C o r o l l. II.

50. Si ex ſingulis abſciſſæ diviſionibus applicatæ ducantur, atque ea quæ abſciſſæ $AM = x$ reſpondet, nempe Mm, ponatur $= y$, reliquæ tam ſequentes quam antecedentes, ita denotabuntur

$$\begin{array}{l|l} Mm = y & Mm = y \\ Nn = y' & Ll = y_{,} \\ Oo = y'' & Kk = y_{,,} \\ Pp = y''' & Ii = y_{,,,} \\ Qq = y^{IV} & Hh = y_{IV} \\ \quad \&c. & \quad \&c. \end{array}$$

Co-

Coroll. III.

51. Cum deinde valor ipſius p ſit $= \frac{dy}{dx} = \frac{Nn - Mm}{dx}$; erit $p = \frac{y' - y}{dx}$; ſequentes autem pariter ac antecedentes ipſius p valores ita ſe habebunt:

$p = \frac{y' - y}{dx}$	$p = \frac{y' - y}{dx}$
$p' = \frac{y'' - y'}{dx}$	$p_{,} = \frac{y - y_{,}}{dx}$
$p'' = \frac{y''' - y''}{dx}$	$p_{,,} = \frac{y_{,} - y_{,,}}{dx}$
$p''' = \frac{y^{IV} - y'''}{dx}$	$p_{,,,} = \frac{y_{,,} - y_{,,,}}{dx}$
&c.	&c.

Coroll. IV.

52. Deinde, quia eſt $q = \frac{dp}{dx} = \frac{p' - p}{dx}$ erit $q = \frac{y'' - 2y' + y}{dx^2}$; ex quo quantitatis q valores, cum ſequentes tum antecedentes, ita ſe habebunt:

$q = \frac{y'' - 2y' + y}{dx^2}$	$q = \frac{y'' - 2y' + y}{dx^2}$
$q' = \frac{y''' - 2y'' + y'}{dx^2}$	$q_{,} = \frac{y' - 2y + y_{,}}{dx^2}$
$q'' = \frac{y^{IV} - 2y''' + y''}{dx^2}$	$q_{,,} = \frac{y - 2y_{,} + y_{,,}}{dx^2}$
&c.	&c.

Coroll. V.

53. Simili igitur modo per iſta applicatarum ſigna poterunt

Digitized by Google

valores quantitatum r, s, t, &c. ut has ſupra aſſumſimus, determinari, atque ex figura definiri. Erit ſcilicet

$$r = \frac{y''' - 3y'' + 3y' - y}{dx^3}$$
$$s = \frac{y^{IV} - 4y''' + 6y'' - 4y' + y}{dx^4}$$
$$t = \frac{y^{V} - 5y^{IV} + 10y''' - 10y'' + 5y' - y}{dx^5}$$
&c.

unde harum litterarum valores tam præcedentes quam antecedentes formari poſſunt.

Coroll. VI.

54. Quod ſi autem formula $\int Z\,dx$ ad abſciſſam $AM = x$ fuerit relata; erit ejus valor ſequenti abſciſſæ elemento $MN = dx$ reſpondens $= Z\,dx$. Hincque ſimili modo formulæ $\int Z\,dx$ valores ſingulis abſciſſæ elementis reſpondentes denotabuntur ut ſequitur:

pro $MN = Z\,dx$	pro $MN = Z\,dx$
pro $NO = Z'dx$	pro $LM = Z_{,}dx$
pro $OP = Z''dx$	pro $KL = Z_{,,}dx$
pro $PQ = Z'''dx$	pro $IK = Z_{,,,}dx$
&c.	&c.

Coroll. VII.

55. Si ergo expreſſio $\int Z\,dx$ ad abſciſſam curvæ $AM = x$ pertineat; ejuſdem expreſſionis valor, qui conveniet abſciſſæ propoſitæ AZ, erit $= \int Z\,dx + Z\,dx + Z'dx + Z''dx + Z'''dx$ + &c. in infinitum, donec perveniatur ad ultimum punctum Z.

Coroll. VIII.

56. Si igitur curva inveniri debeat, quæ pro data abſciſſa AZ valo-

valorem formulæ $\int Z dx$ habeat maximum minimumve; tum, poſita abſciſſa quacunque indefinita $AM = x$, efficiendum eſt ut hæc expreſſio $\int Z dx + Z dx + Z' dx + Z' dx + Z'' dx +$ &c. uſque in Z fiat maxima vel minima.

SCHOLION.

57. Quanquam hæc hypotheſis tantum pro arbitrio eſt facta; tamen iſta ſigna maximam afferent utilitatem ad Problemata, quæ ad hanc methodum maximorum & minimorum pertinent, ſuccincte reſolvenda. Plurimum enim valet in hujuſmodi negotiis commoda ſignorum electio, ejuſque ope calculus non ſolum contrahi, ſed etiam multo facilior & expeditior reddi poteſt. Præſtabit autem iſte ſignandi modus longe alteri recepto, quo per differentialia valores functionum variabilium proxime ſequentes exprimi ſolent; eo quod in ipſa reſolvendi methodo alius generis differentialia occurrent, quæ cum naturalibus quantitatum variabilium differentialibus facile confundi poſſent, niſi, iſta aſſumta ſignandi methodo o naturalia differentialia notatione ſaltem tollerentur.

PROPOSITIO IV. THEOREMA.

58. *Si* amnoz *fuerit curva ad abſciſſam datam* AZ *relata, in qua formula* $\int Z dx$ *maximum minimumve obtineat valorem; atque alia concipiatur curva* amνoz *ab iſta infinite parum diſcrepans, tum valor formulæ* $\int Z dx$ *pro utraque curva erit idem.* *Fig. 3.*

DEMONSTRATIO.

Quando in Analyſi formula quæpiam variabilis fit maxima; tum primo creſcendo continuo magis ad maximum valorem accedit, deinde vero cum hunc attigit, iterum decreſcendo ab eo recedit. Iſte autem acceſſus ad maximum valorem atque receſſus ab eodem ita fit, ut dum quantitas proxime ad maximum valorem verſatur, tum ejus incrementa ac decrementa momenta-

Digitized by Google

nea evaneſcant; hocque idem de minimo eſt intelligendum. Dantur quidem etiam ejuſmodi maxima & minima, circa quæ incrementa & decrementa ſint infinite magna; verum hujus generis maxima & minima in præſenti inſtituto raro locum inveniunt, & ſi inveniunt, facile erit ea determinare. Sufficiat igitur notaſſe circa maximum & minimum mutationes momentaneas non dari poſſe finitas. Quod ſi ergo in curva a m n o z expreſſio $\int Z\,dx$ maximum minimumve habeat valorem; pro alia curva ejuſdem expreſſionis valor eo magis a maximo minimove recedet, quo magis hæc alia curva ab illa diſcrepet. Sin autem alia curva infinite parum differat ab illa ſatisfaciente, tum, pro utraque, formula $\int Z\,dx$ eundem obtinebit valorem. Hujuſmodi autem curvam minime diſcrepantem concipiemus, ſi arcum tantum infinite parvum m n o infinite parum variari, ejuſque loco arcum m ν o ſubſtitui ponamus. Quamobrem ex curva a z, pro qua $\int Z\,dx$ maximum eſt vel minimum, portionem infinite parvam m n o exſcindi, ejuſque loco aliam m ν o infinite parum ab illa diſcrepantem inſeri intelligamus; tum valor formulæ $\int Z\,dx$ qui convenit curvæ a m n o z æqualis erit valori, qui convenit curvæ a m ν o z. *Q. E. D.*

COROLL. I.

59. Quoniam mutatio debet poni quamminima; non ſufficiet arcum m n o, qui immutari ponitur, accipere infinite parvum, ſed etiam deviatio n ν, præ arcus longitudine m n o, debet eſſe infinite parva.

COROLL. II.

60. Poſita igitur tali mutatione in curva, mutatio inde etiam in valore formulæ $\int Z\,dx$ orietur; quæ autem per demonſtrationem erit evaneſcens. Atque hoc modo ex tali aſſumta mutatione orietur æquatio, quæ ſimul curvæ quæſitæ naturam præbebit.

SCHO-

Digitized by Google

SCHOLION.

61. In hac Propoſitione continetur univerſa methodus reſolvendi Problemata, quibus curva deſideratur in qua valor formulæ cujuſdam indeterminatæ ut $\int Z dx$ ſit maximus vel minimus. Semper enim concipitur portio curvæ infinite parva, uti m n o, aliquantillum variari in m ν o, atque tum quæritur differentia valorum quos formula $\int Z dx$, cum pro curva vera a m n o z, tum pro ficta a m ν o z, ſortitur, eaque differentia nihilo æqualis poſita dat naturam curvæ quæſitæ. Mutatio autem iſta in loco indefinito fieri debet, ut ad totam curvam pertineat, atque ad ſingula loca pateat. Poteſt autem iſta mutatio utcunque inſtitui, dummodo ſit infinite parva, atque vel ad duo vel plura curvæ elementa extendi; ſemper enim eadem reſultare debet æquatio finalis. Interim tamen calculi commoditas poſtulat, ut mutatio in tam paucis elementis inſtituatur, quæ ſufficiat ad ſolutionem abſolvendam. Ita ſi, inter omnes omnino curvas eidem abſciſſæ reſpondentes, ea determinari debeat, in qua ſit $\int Z dx$ maximum vel minimum; tum ſufficiet bina tantum curvæ elementa mutata concipere. At ſi non inter omnes curvas, ſed eas tantum quæ unam pluresve expreſſiones communes habeant, ea definiri debeat in qua quæpiam quantitas ſit maxima vel minima; tum mutationem non quamcunque m ν o accipere licet, ſed talem ſtatui oportet, ut illæ proprietates omnibus curvis communes conſerventur. His igitur caſibus, duo elementa non ſufficient, ſed plura accipi debebunt, ut omnibus conditionibus ſatisfieri queat.

DEFINITIO V.

62. *Valor Differentialis* datæ maximi minimive formulæ reſpondens eſt differentia inter valores, quos hæc formula, cum in ipſa curva quæſita, tum in eadem infinite parum immutata, obtinet.

Digitized by Google

C o r o l l. I.

63. In curva igitur, pro qua data formula, puta $\int Zdx$, maximum minimumve eſſe debet, hujus formulæ valor differentialis reſpondens evaneſcet. Atque hanc ob rem ſi valor differentialis nihilo æqualis ponatur; habebitur æquatio, qua curvæ quæſitæ natura exprimetur.

C o r o l l. II.

64. Ex invento igitur valere differentiali, qui propoſitæ maximi minimive formulæ reſpondeat, ſtatim habebitur æquatio exprimens naturam ejus curvæ, in qua formula illa propoſita maximum minimumve habeat valorem.

C o r o l l. III.

65. Totum igitur negotium ad curvas inveniendas, quæ maximi minimive proprietate gaudeant, eo eſt reductum, ut pro quaque maximi minimive formula ejus conveniens valor differentialis inveſtigetur.

S C H O L I O N.

66. Cum igitur in genere tradita ſit idea non ſolum naturæ quæſtionum, quibus curvæ maximi minimive proprietate præditæ quæruntur, ſed etiam methodi, qua ad eas reſolvendas uti oporteat, ad ipſam tractationem progrediemur. Ac primo quidem Methodum abſolutam, qua curvæ quæruntur quæ inter omnes omnino curvas ad eandem abſciſſam relatas maximi minimive proprietate quapiam ſint præditæ, trademus. Deinde pergemus ad Methodum maximorum ac minimorum relativam, ad quam tales pertinent quæſtiones, quæ non inter omnes curvas datæ abſciſſæ reſpondentes, ſed eas tantum quæ data quadam communi proprietate una pluribuſve gaudent, eam determinari jubent, cui maximi minimive prærogativa quæpiam conveniat.

Digitized by Google

niat. In has autem tractationes natura formulæ $\int Z\,dx$, quæ maximum minimumve eſſe debet, ingens diſcrimen infert, prout Z fuerit functio vel determinata vel indeterminata; quemadmodum jam obſervavimus.

CAPUT II.

De Methodo maximorum ac minimorum ad lineas curvas inveniendas abſoluta.

PROPOSITIO I. PROBLEMA.

1. *SI in curva quacunque* amz *una applicata quævis* Nn *augeatur particula infinite parva* nν; *invenire incrementa vel decrementa, quæ ſingulæ quantitates determinata ad curvam pertinentes hinc accipient.* Fig. 4.

SOLUTIO.

Quantitates determinatæ ad curvam propoſitam pertinentes ſunt, præter abſciſſam x, quæ non afficitur, hæ y, p, q, r, s, &c. cum ſuis derivatis valoribus, quos in locis vel ſequentibus vel antecedentibus ſortiuntur. Quod ſi nunc ponamus AM $= x$, & Mm $= y$, erit Nn $= y'$, hujuſque valor per translationem puncti n in ν augebitur particula nν, reliquæ autem applicatæ y'', y''', y^{IV}, &c. pariter ac præcedentes $y_{,}$, $y_{,,}$, $y_{,,,}$, $y_{,v}$ &c. non afficientur. Cum igitur ſola applicata y' creſcat particula nν; ex Cap. præc. §. §. 51 & ſeqq. colligetur quantum incrementum reliquæ quantitates omnes capiant ex incremento ſolius applicatæ y'. Omnes ſcilicet quantitates, quarum valor pendet ab y', mutationem ſubibunt, reliquæ vero, quæ ab y' non pendent, manebunt invariatæ. Ita cum ſit $p = \frac{y' - y}{dx}$ hæc quantitas p creſcet particula $\frac{n\nu}{dx}$; at cum ſit $p' = \frac{y'' - y'}{dx}$, hæc quantitas p' de-

Digitized by Google

p' decreſcet particula $\frac{n\nu}{dx}$. Similique modo reliquarum quantitatum incrementa vel decrementa reperientur, delendo in earum valoribus ſupra exhibitis omnes valores ipſius y, præter hunc y', hujuſque loco ſcribendo $n\nu$. Hoc modo omnium quantitatum determinatarum, quæ quidem mutationem patiuntur, incrementa in ſequenti Tabella congeſſimus

Quant.	*Increm.*	*Quant.*	*Increm.*
y'	$+ n\nu$	$s_{,,,}$	$+ \frac{n\nu}{dx^4}$
p	$+ \frac{n\nu}{dx}$	$s_{,,}$	$- \frac{4n\nu}{dx^4}$
p'	$- \frac{n\nu}{dx}$	$s_{,}$	$+ \frac{6n\nu}{dx^4}$
$q_{,}$	$+ \frac{n\nu}{dx^2}$	s	$- \frac{4n\nu}{dx^4}$
q	$- \frac{2n\nu}{dx^2}$	s'	$+ \frac{n\nu}{dx^4}$
q'	$+ \frac{n\nu}{dx^2}$	$t_{,,,,}$	$+ \frac{n\nu}{dx^5}$
$r_{,,}$	$+ \frac{n\nu}{dx^3}$	$t_{,,,}$	$- \frac{5n\nu}{dx^5}$
$r_{,}$	$- \frac{3n\nu}{dx^3}$	$t_{,,}$	$+ \frac{10n\nu}{dx^5}$
r	$+ \frac{3n\nu}{dx^3}$	$t_{,}$	$- \frac{10n\nu}{dx^5}$
r'	$- \frac{n\nu}{dx^3}$	t	$+ \frac{5n\nu}{dx^5}$
		t'	$- \frac{n\nu}{dx^5}$

Atque ex hac Tabella etiam ulteriorum quantitatum, ſi quæ occurrunt, incrementa vel decrementa facile cognoſci poterunt, Q. E. I.

C o-

COROLL. I.

2. Cognitis igitur incrementis harum quantitatum primariarum ad curvam pertinentium, inde omnium quantitatum ex iis compoſitarum incrementa, quæ oriuntur ex aucta applicata y', determinari poterunt, ſi ratio compoſitionis ſpectetur.

COROLL. II.

3. Harum ſcilicet quantitatum incrementa exhibita, conſiderari poterunt tanquam earum differentialia. Atque ſi propoſita fuerit quantitas quæcunque ex illis compoſita, ejus conveniens incrementum ex tranſlatione puncti n in ν ortum invenietur, differentiando illam quantitatem, & loco differentialium ſingularum quantitatum, ſcribendo ea incrementa, quæ his quantitatibus ſunt adſcripta.

COROLL. III.

4. Si igitur habeatur hæc functio $y'\surd(1+pp)$, cujus incrementum, quod ex tranſlatione puncti n in ν oritur ſit determinandum; ea functio primum differentietur; unde prodibit $dy'\surd(1+pp)+\frac{y'pdp}{\surd(1+pp)}$; hicque loco dy' & dp ſcribantur incrementa quantitatibus y' & p convenientia, nempe $+n\nu$ & $+\frac{n\nu}{dx}$; eritque functionis propoſitæ incrementum $=+n\nu.\surd(1+pp)+\frac{y'p.n\nu}{dx\surd(1+pp)}$.

COROLL. IV.

5. Expedite igitur per differentiationem functionis cujuſcunque, incrementum, quod ex incremento nν applicatæ y' oritur, aſſignari poteſt; id quod ex inſpectione figuræ difficulter & minime generaliter fieri poteſt.

Digitized by Google

S C H O L I O N.

6. Probe notandum eſt hunc modum incrementa functionum ſeu quantitatum ex x, y, p, q, &c. harumque derivatis y', y'', p', p'', &c. datarum incrementa inveniendi, tantum ad functiones determinatas patere, minime vero ad indeterminatas extendi poſſe. Quod ſi enim functio propoſita fuerit indeterminata, ſeu formula integralis indefinita, integrationem neque algebraïce neque tranſcendenter admittens, tum differentiatione nihil conſequimur ad ejus incrementum inveniendum. In ſequentibus autem, ubi ejuſmodi maximi minimive formulas $\int Z\,dx$ ſumus contemplaturi, in quibus Z ſit functio talis indeterminata, in hujuſmodi functionum incrementa ſumus inquiſituri. Sin autem Z fuerit functio determinata, propoſiti Problematis ſolutio ſufficere poteſt ad ſolutiones Problematum huc pertinentium abſolvendas.

Propositio II. Problema.

Fig. 4. 7. *Si fuerit* Z *functio determinata ipſarum* x *&* y *tantum, invenire curvam* az, *in qua valor formulæ* $\int Z\,dx$ *ſit maximus vel minimus.*

S O L U T I O.

Concipiatur abſciſſa AZ, cui maximum minimumve formulæ $\int Z\,dx$ reſpondere debet, diviſa in innumerabilia elementa æqualia, ſingula per dx denotanda; poſitaque abſciſſa indefinita AM $= x$, & applicata Mm $= y$, ex formula $\int Z\,dx$ elemento MN reſpondebit $Z\,dx$; atque ſecundum receptum notandi modum, elemento ſequenti NO reſpondebit $Z'\,dx$, & ſequentibus elementis OP, PQ &c. reſpondebunt valores $Z''dx$, $Z'''dx$, &c. antecedentibus vero elementis LM, KL, IK, reſpondebunt $Z_{,}dx$; $Z_{,,}dx$; $Z_{,,,}dx$, &c. Quare ſi curva az ſit ea ipſa, quæ quæritur, debebit eſſe $Z\,dx + Z'\,dx + Z''\,dx +$ &c. una cum

$Z_{,}dx$

Digitized by Google

$Z_{,}dx + Z_{,,}dx + Z_{,,,}dx +$ &c. maximum vel minimum. Quod ſi igitur una applicata $Nn = y'$ augeatur particula $n\nu$, illa expreſſio eundem valorem retinere, atque adeo valor differentialis formulæ $\int Zdx$, ſeu ſummæ terminorum $Zdx + Z'dx + Z''dx + Z'''dx +$ &c. una cum $Z_{,}dx + Z_{,,}dx + Z_{,,,}dx +$ &c. evaneſcere debet. Singulorum igitur horum terminorum valores differentiales, qui oriuntur ex translatione puncti n in ν, inveſtigari debebunt; eorumque aggregatum erit valor differentialis formulæ $\int Zdx$ reſpondens, qui poſitus $= 0$ æquationem pro curva quæſita præbebit. Quoniam autem Z ponitur functio determinata ipſarum x & y; habebit ipſius differentiale dZ hujuſmodi formam $Mdx + Ndy$; ita ut ſit $dZ = Mdx + Ndy$. Valorum igitur derivatorum ipſius Z differentialia ita ſe habebunt.

$dZ' = M'dx + N'dy'$	$dZ_{,} = M_{,}dx + N_{,}dy_{,}$
$dZ'' = M''dx + N''dy''$	$dZ_{,,} = M_{,,}dx + N_{,,}dy_{,,}$
&c.	&c.

Cum nunc valores differentiales terminorum Zdx, $Z'dx$, $Z''dx$, &c. itemque ipſorum $Z_{,}dx$, $Z_{,,}dx$, &c. inveniantur, ſi hi termini differentientur, atque loco dy' in differentialibus ſcribatur $n\nu$, loco omnium reliquorum differentialium vero o; manifeſtum eſt ſolum terminum $Z'dx$ habiturum eſſe valorem differentialem, quoniam in ejus ſolius differentiali occurrit dy'. Scripto itaque $n\nu$ loco dy', erit termini $Z'dx$ valor differentialis $= N'dx.\, n\nu$, qui ſimul erit valor differentialis totius formulæ $\int Zdx$; quia reliqui termini præter $Z'dx$ nullam variationem patiuntur. Loco N' autem ponere poterimus N, quia eſt $N' = N + dN$, & dN præ N evaneſcit. Pro curva igitur quæſita, in qua ſit $\int Zdx$ maximum vel minimum, iſta habetur æquatio $Ndx.\, n\nu = 0$ ſeu $N = 0$; exiſtente $dZ = Mdx + Ndy$. *Q. E. I.*

Digitized by Google

COROLL. I.

8. Si igitur curva debeat definiri, in qua ſit $\int Z\,dx$ maximum vel minimum, atque Z ſit functio determinata ipſarum x & y tantum; tum quantitatem Z differentiari oportet; quod cum habiturum ſit hujusmodi formam $dZ = M\,dx + N\,dy$, hinc formabitur æquatio pro curva quæſita, quæ erit $N = 0$.

COROLL. II.

9. Cum ergo N ſit functio ipſarum x & y determinata, in æquatione pro curva $N = 0$ nulla inerit quantitas conſtans, quæ non fuit in formula maximi minimive $\int Z\,dx$; & hanc ob rem curva inventa erit unica & perfecte determinata.

COROLL. III.

10. In quæſtionibus igitur ſub hoc Problemate comprehenſis, curva ſatisfaciens ex ſola maximi minimive formula determinatur; neque licebit inſuper puncta aliqua præſcribere, per quæ curva quæſita tranſeat.

COROLL. IV.

11. Quod ſi Z fuerit functio tantum ipſius x, ita ut y non involvat; erit tum $\int Z\,dx$ functio determinata pariter ipſius x tantum; eique adeo omnes curvæ eidem abſciſſæ reſpondentes æque ſatisfacient. Idem vero hoc monſtrat calculus; hoc enim caſu, quo in Z non ineſt y, fiet $N = 0$; ideoque nulla prodit æquatio pro curva quæſita.

COROLL. V.

12. Statim etiam intelligi poteſt, utrum detur linea curva; in qua hujuſmodi formula $\int Z\,dx$ ſit maximum vel minimum. Si enim ex differentiatione ipſius Z ejuſmodi valor pro N reperiatur;

tur;

Digitized by Google

tur, ut per æquationem $N = 0$ nulla curva exprimatur; tum etiam nulla curva extat in qua propoſita formula $\int Z\,dx$ ſit maximum vel minimum.

Coroll. VI.

13. Denique etiam perſpicitur, hanc maximi minimive proprietatem non uni alicui determinatæ abſciſſæ eſſe adſtrictam; ſed ſi curva pro una abſciſſa reddat formulam $\int Z\,dx$ maximum vel minimum, eandem pro quacunque alia abſciſſa, pariter maximum minimumve valorem eſſe habiturum.

SCHOLION I.

14. Nacti ergo ſumus methodum facilem, inter omnes curvas eidem abſciſſæ reſpondentes, eam determinandi, in qua conſtituat formula $\int Z\,dx$ valorem maximum vel minimum, ſiquidem Z eſt functio determinata ipſarum x & y tantum. Simul vero etiam patet curvam ſatisfacientem ſemper fore algebraïcam, ſiquidem Z fuerit functio algebraïca ipſarum x & y. Curvæ igitur hoc modo inventæ iſta erit proprietas, ut ſi ad eandem abſciſſam alia quæcunque cônſtituatur linea curva, tum pro ea valor formulæ $\int Z\,dx$ certo vel minor vel major ſit proditurus quam pro inventa; prout in inventa formula $\int Z\,dx$ vel fuerit maxima vel minima. Cum autem adhuc dubium ſit utrum in curva inventa valor formulæ $\int Z\,dx$ futurus ſit maximus an minimus; de eo in quovis caſu particulari facile fiet dijudicatio; in genere autem nihil omnino decidi poteſt. Interim hoc certum eſt, ſi unica prodit æquatio, tum tantum vel maximum vel mimum locum habere poſſe; hoc eſt, ſi curva inventa ſit pro maximo; tum minimum non dari, ſed valorem formulæ $\int Z\,dx$ in infinitum diminui poſſe. Pari modo, ſi unica inventa fuerit curva, in eaque formula $\int Z\,dx$ ſit minima, tum valorem $\int Z\,dx$ in infinitum augeri poſſe. Quod ſi autem ſolutio nullam prorſus præbeat curvam ſatisfacientem, id indicio erit valorem formulæ

Digitized by Google

$\int Z\,dx$ pro quacunque abſciſſa tam in infinitum creſcere quam decreſcere poſſe.

SCHOLION II.

15. Ex eadem etiam ſolutione reperiri poterunt illæ curvæ maximi minimive proprietate præditæ alterius generis ſupra memoratæ, ad quas non pervenitur per valores differentiales evaneſcentes, ſed infinite magnos; quod maximorum & minimorum genus ab illo maxime diſcrepat. Reperientur autem iſtæ curvæ, ſi valor differentialis $N\,dx.\ n$, non nihilo, ſed infinito æqualis ponatur. Quoties igitur hæc æquatio $N = \infty$ lineam aliquam curvam ſuggerit; tum in ea pariter formula $\int Z\,dx$ maximum vel minimum obtinebit valorem: Hoc ſcilicet eveniet, quando pro N prodit fractio, cujus denominator nihilo æqualis poſitus, præbet æquationem pro aliqua linea curva. Hoc itaque pacto plures curvæ reperiri poſſunt, quæ eidem quæſtioni ſatisfaciant; quarum aliæ maxima continebunt, aliæ minima. Fieri etiam poteſt, ut plures quam duæ curvæ Problemati ſatisfacientes reperiantur, etiamſi binæ tantum oriri queant æquationes, ſcilicet $N = 0$ & $N = \infty$. Si enim N fuerit quantitas ex factoribus compoſita; tum quilibet factor, vel nihilo vel infinito æqualis poſitus, dabit æquationem pro curva ſatisfaciente; conſtat enim ſæpenumero plura maxima pluraque minima locum habere poſſe. Hæc autem omnia clarius enodabuntur in ſequentibus Exemplis in hoc Problemate contentis.

EXEMPLUM I.

16. *Invenire curvam, quæ, inter omnes omnino curvas eidem abſciſſæ reſpondentes, habeat* $\int XY\,dx$ *maximum vel minimum; denotante* X *functionem ipſius* y, *&* Y *ipſius* y *tantum.*

In hoc igitur caſu fiet $Z = XY$; ideoque $dZ = Y\,dX + X\,dY = M\,dx + N\,dy$. Erit ergo $M = \frac{Y\,dX}{dx}$ & $N = \frac{X\,dY}{dy}$;

&

ob X ipſius x & Y ipſius y functionem. Pro curva igitur quæſita erit $N = \frac{X\,dY}{dy} = 0$: quoniam autem Y eſt functio ipſius y, ponatur $dY = \Theta\,dy$; erit Θ pariter functio ipſius y; ideoque pro curva quæſita, ſi quæ ſatisfacit, habetur hæc aquatio $X\Theta = 0$, ideoque vel $X = 0$, vel $\Theta = 0$; quarum cum neutra lineam curvam præbeat, apparet huic quæſtioni nullam omnino curvam ſatisfacere, ſed valorem propoſitum $\int XY\,dx$ in infinitum cum augeri tum diminui poſſe. Ex æquatione autem $\Theta = 0$, quia Θ eſt functio ipſius y, ſequitur $y =$ *Conſt.* quæ æquatio præbet lineam rectam parallelam abſciſſæ AZ, cujus diſtantia tanta eſt, ut fiat functio Y maxima vel minima. Patet enim, ſi quantitas Y maximum minimumve valorem admittat, tum etiam formulam $\int XY\,dx$ fieri maximum vel minimum. Altera autem æquatio $X = 0$, quia præbet $x =$ *Conſt.* nequidem lineam rectam quæſtioni ſatisfacientem exhibet; quia præbet lineam rectam normalem ad abſciſſam, quæ propterea non datæ abſciſſæ cuipiam, ſed tantum ejus uni puncto reſpondebit.

EXEMPLUM II.

17. *Invenire curvam, quæ, inter omnes eidem abſciſſæ reſpondentes curvas, habeat valorem formulæ* $\int(ax - yy)\,y\,dx$ *maximum vel minimum.*

Si hæc formula cum generali $\int Z\,dx$ compareretur, fiet $Z = axy - y^3$, ideoque $dZ = ay\,dx + (ax - 3yy)\,dy$; ita ut fiat $M = ay$ & $N = ax - 3yy$; unde pro curva quæſita habebitur iſta æquatio $ax - 3yy = 0$, ſeu $yy = \frac{1}{3}ax$, quæ eſt pro Parabola verticem in A, axem AZ & parametrum $= \frac{1}{3}a$ habente. In hac igitur Parabola, erit valor formulæ $\int(ax - yy)\,y\,dx$ maximus vel minimus. Utrum autem ſit maximus an minimus, reperietur, ſi aliam quamcunque lineam loco Parabolæ ſubſtituamus, atque inquiramus utrum pro ea valor formulæ propoſitæ major ſit an minor quam pro Parabola. Sumamus igitur

igitur lineam rectam cum ipſo axe congruentem, pro qua erit $y = 0$. Pro hac itaque valor formulæ $\int (ax - yy) y\,dx$ fiet pariter $= 0$, pro Parabola autem idem valor erit affirmativus, ideoque > 0; ex quo ſequitur in Parabola formulæ propoſitæ valorem non eſſe minimum, ſed maximum. Poterimus autem algebraïce indicare quantus futurus ſit valor formulæ propoſitæ pro Parabola: cum enim ſit $yy = \frac{1}{3} ax$, abibit formula propoſita in hanc $\int \frac{2}{3} ax\,dx \sqrt{\frac{1}{3} ax} = \frac{4}{15} ax^2 \sqrt{\frac{1}{3} ax}$. Quod ſi autem ponamus aliam æquationem, puta $y = nx$; abibit formula propoſita in hanc $\int dx (naxx - n^3x^3) = \frac{1}{3} nax^3 - \frac{1}{4} n^3x^4$, quæ ſemper eſt minor quam valor formulæ qui pro Parabola inventa prodiit: id quod quilibet facile, ſubſtituendis loco x definitis valoribus, experietur.

Exemplum III.

18. *Invenire curvam, in qua ſit, inter omnes omnino curvas ad eandem abſciſſam relatas, valor hujus formulæ* $\int (15a^2x^2y - 15a^3xy + 5a^2y^3 - 3y^5)\,dx$ *maximus vel minimus.*

Erit igitur $Z = 15a^2x^2y - 15a^3xy + 5a^2y^3 - 3y^5$, qui ſi differentietur, poſito x conſtante, prodibit $N\,dy = 15a^2x^2\,dy - 15a^3x\,dy + 15a^2y^2\,dy - 15y^4\,dy$; hincque $N = 15(a^2x^2 - a^3x + a^2y^2 - y^4)$; qui valor, poſitus $= 0$, dabit æquationem pro curva quæſita: erit itaque $aaxx - a^3x + a^2y^2 - y^4 = 0 = (ax - yy)(ax + yy - aa)$. Ob binos hos factores, prodeunt binæ curvæ ſatisfacientes, quarum altera exprimetur hac æquatione $yy = ax$, altera hac $yy = aa - ax$; utraque pro Parabola. Ut nunc appareat utra ſit pro maximo vel minimo, ponamus abſciſſam eſſe minimam, ac prior æquatio $yy = ax$ in formula ſubſtituta dabit $\int -10a^3x\,dx \sqrt{ax}$. Altera vero formula $yy = aa - ax$, ſeu $y = a$, ſubſtituta dabit $\int 2a^5\,dx$. Quod ſi autem ipſi y alius quicunque valor tribuatur, puta $y = 0$; tum formula propoſita abit in $\int 0\,dx = 0$. Ex quo patet curvarum inventarum alteram $yy = aa - ax$ eſſe pro maximo, alteram autem $yy = ax$ pro minimo, ſcilicet pro maximo negativo.

tivo. Facillime autem perpetuo hæc dijudicatio, utrum maximum an minimum in curva inventa locum habeat, inſtituetur, ſi abſciſſa x ponatur infinite parva; tum enim integratione non erit opus, ſed ipſa formula Zdx monſtrabit valorem formulæ $\int Zdx$ hoc caſu.

EXEMPLUM IV.

19. *Inter omnes curvas eidem abſciſſæ reſpondentes, definire eam in qua ſit formula* $\int(3ax - 3xx - yy)(ax - xx - \frac{4}{3}xy + yy)\,dx$ *valor maximus vel minimus.*

Ex hac igitur formula prodibit ſequens ipſius Z valor evolutus;

$$Z = \begin{array}{l} +\,3a^2x^2 - 4ax^2y + 2axyy + \frac{4}{3}xy^3 - y^4 \\ -\,6ax^3 + 4x^3y - 2xxyy \\ +\,3x^4 \end{array}$$

quæ differentiata, poſito x conſtante, ac diviſa per dy, ſequentem præbebit valorem pro N:

$$N = \begin{array}{l} -\,4ax^2 + 4axy + 4xyy - 4y^3 \\ +\,4x^3 - 4xxy \end{array}$$

quæ expreſſio, nihilo æqualis poſita, dabit æquationem pro curva quæſita. Erit itaque

$$\begin{array}{l} y^3 - xyy + xxy + axx = 0 \\ \quad\quad - axy - x^3 \end{array}$$

quæ duos habet factores, qui totidem præbent æquationes; haſce

I. $y - x = 0$, pro linea recta
II. $yy - ax + xx = 0$, pro circulo.

Ponatur x infinite parva, eritque ex æquatione $y = x$, valor ipſius $Z = 3a^2x^2$; at ex æquatione $yy = ax - xx$, ſeu $y = \sqrt{ax}$, erit $Z = 4aaxx$. Quod ſi autem ponatur $y = a$, prodit $Z = -a^4$, unde apparet utramque lineam inventam eſſe pro maximo.

Digitized by Google

SCHOLION

20. Problemata etiam reſolvi poſſunt per Methodum maximorum & minimorum vulgarem. Quando enim curva quæritur pro qua valor ipſius $\int Z dx$ ſit maximus vel minimus, idque pro qualibet abſciſſa; manifeſtum eſt ſiquidem Z ſit functio determinata ipſarum x & y, formulam $\int Z dx$ maximum minimumve eſſe non poſſe, niſi elementum ejus $Z dx$ ac proinde ipſum Z tale ſit. Quamobrem quæſtioni ſatisfiet, ſi quantitas Z differentietur poſito x conſtante, ejuſque differentiale ponatur $= 0$. Tum enim perpetuo Z habebit valorem maximum vel minimum, ac proinde etiam $Z dx$ & ipſa formula $\int Z dx$. Quod ſi autem functio Z differentietur, poſito x conſtante, prodibit $N dy$; quoniam generaliter differentiando poſuimus $dZ = M dx + N dy$; ſatisfietque ponendo $N = 0$: quæ eſt eadem ſolutio, quam per Methodum traditam invenimus. Quamvis autem hinc videantur iſtæ quæſtiones ſimili modo reſolvi poſſe, quo in Methodo maximorum & minimorum vulgari; tamen hoc tantum evenit, ſi Z fuerit functio ipſarum x & y tantum; namque ſi in Z præterea inſint quantitates ex differentialibus ortæ p, q, r, &c: tum vulgaris Methodus nullius amplius uſus eſſe poteſt. Etſi enim tum differentietur functio Z poſito x conſtante, tamen in differentiale etiam ingrederentur differentialia dp, dq, dr, &c. quorum relatio ad dy cum non conſtet, æquatio inde ad maximum minimumve determinandum apta deduci non poterit. His igitur caſibus utilitas & neceſſitas noſtræ Methodi maxime cernetur.

PROPOSITIO III. PROBLEMA.

Fig. 4. 21. *Si* Z *fuerit functio ipſarum* x, y, *&* p *determinata, ita ut ſit* $dZ = M dx + N dy + P dp$; *invenire, inter omnes curvas eidem abſciſſæ reſpondentes, eam in qua ſit* $\int Z dx$ *maximum vel minimum.*

SOLUTIO.

SOLUTIO.

Sit amz curva quæſito ſatisfaciens, atque concipiatur applicata quæcunque $Nn = y'$ augeri particula $n\nu$, debebit valor differentialis formulæ $\int Z\,dx$, ſeu quantitatis huic æquivalentis, puta $Z\,dx + Z'\,dx + Z''\,dx + Z'''\,dx +$ &c. una cum $Z_{,}dx + Z_{,,}dx + Z_{,,,}dx +$ &c. eſſe $= 0$. Totius igitur quantitatis $\int Z\,dx$ valor differentialis ex translatione puncti n in ν habebitur, ſi ſingulorum illorum terminorum, qui quidem hac translatione afficiuntur, valores differentiales quærantur & in unam ſummam addantur. Ex translatione autem puncti n in ν, illi tantum termini mutationem ſubeunt, in quibus inſunt quantitates y', p & p'; ideoque tantum termini $Z\,dx$ & $Z'\,dx$; nam uti Z eſt functio ipſarum y & p præter x; ita Z' ſimilis eſt functio ipſarum y' & p'. Quamobrem hi termini debebunt differentiari, atque in eorum differentialibus loco dy', dp, & dp' ſcribi oportet valores ſupra indicatos $+ n\nu$; $+ \frac{n\nu}{dx}$ & $- \frac{n\nu}{dx}$. Sicut autem eſt $dZ = M\,dx + N\,dy + P\,dp$, ita erit $dZ' = M'dx + N'dy' + P'dp'$. Hinc itaque valor differentialis ipſius Z erit $P.\frac{n\nu}{dx}$, & ipſius Z' erit $N'.n\nu - P'.\frac{n\nu}{dx}$; ex quo utriuſque termini $Z\,dx + Z'\,dx$, ideoque integræ formulæ $\int Z\,dx$ valor differentialis erit $= n\nu.(P + N'dx - P')$. At eſt $P' - P = dP$, & loco N' ſcribi poteſt N; unde valor differentialis erit $= n\nu.(N\,dx - dP)$. Quare cum formulæ $\int Z\,dx$ valor differentialis nihilo æqualis factus præbeat æquationem pro curva quæſita, hæc erit $0 = N\,dx - dP$, vel $N - \frac{dP}{dx} = 0$, qua æquatione natura curvæ quæſitæ exprimetur. *Q. E. I.*

COROLL. I.

22. Quod ſi ergo fuerit Z functio quæcunque ipſarum x, y, itemque earum differentialium dx & dy, ſeu loco horum differentialium,

 ipſius

ipſius p; exiſtente $dy = p\,dx$; differentiale ipſius Z hujuſmodi habebit formam, ut ſit $dZ = M\,dx + N\,dy + P\,dp$. Atque hinc reperietur curva, in qua ſit $\int Z\,dx$ maximum vel minimum, formando hanc æquationem $N - \frac{dP}{dx} = 0$ ſeu $N\,dx = dP$.

Coroll. II.

23. Æquatio hæc igitur ſemper erit differentialis ſecundi gradus, niſi in P plane non inſit p. Nam ſi p continetur in P, tum in dP inerit dp; quod ob $p = \frac{dy}{dx}$ differentialia ſecundi gradus involvet.

Coroll. III.

24. Quando ergo in differentiali ipſius $dZ = M\,dx + N\,dy + P\,dp$ quantitas P adhuc in ſe complectitur p; tum, ob æquationem pro curva quæſita differentialem ſecundi gradus, duæ novæ conſtantes arbitrariæ per integrationem ingredientur. Ex quo ad harum conſtantium determinationem, duo curvæ puncta præſcribi poterunt; alias enim non una ſed innumerabiles curvæ reperirentur.

Coroll. IV.

25. Ut itaque hujuſmodi Problemata determinate proponantur, ita ſunt enuncianda, ut per data duo puncta curva duci debeat, quæ, inter omnes alias curvas per eadem puncta ductas, pro eadem abſciſſa x valorem $\int Z\,dx$ maximum minimumve complectatur.

Coroll. V.

26. In P autem quantitas p non inerit, ſi Z fuerit functio ipſarum x & y tantum, per p vel per $n+p$, denotante n numerum

rum conſtantem, multiplicata. Sit enim V functio ipſarum x & y tantum; ita ut ſit $dV = Mdx + Ndy$; atque $Z = V(n+p)$, erit $dZ = (n+p)Mdx + (n+p)Ndy + Vdp$. Hincque æquatio pro curva quæſita erit $o = (n+p)N - \frac{dV}{dx}$, ſeu $(n+p)Ndx = dV = Mdx + Ndy$.

C o r o l l. VI.

27. His igitur caſibus, quibus eſt $Z = V(n+p)$, exiſtente V functione ipſarum x & y tantum, non pervenitur ad æquationem differentialem ſecundi gradus: quia dp in ea prorſus non ineſt. Verum nequidem ad differentialem æquationem primi gradus pervenitur; ſed adeo ad algebraïcam. Nam cum ſit $pdx = dy$, erit $(n+p)Ndx = nNdx + Ndy$; quod ipſi $Mdx + Ndy$ æquale poſitum, dabit æquationem per dx diviſibilem, adeoque algebraïcam, hanc $nN = M$, ſiquidem V fuerit functio algebraïca.

C o r o l l. VII.

28. Quoties autem hoc evenit, maximi minimive formula, quæ eſt $\int Zdx$, erit talis formæ, $\int(Vndx + Vdy)$, vel poſito $n = o$, talis $\int Vdy$. Hujuſmodi igitur maximi minimive formulæ pariter ad æquationem determinatam pro curva quæſita deducunt, ita ut non liceat unum plurave puncta præſcribere, per quæ curva tranſire debeat.

C o r o l l. VIII.

29. Poſita igitur V functione ipſarum x & y, iſta maximi minimive formula $\int Vdy$ pari modo tractatur, quo $\int Vdx$. Nam, poſito $dV = Mdx + Ndy$, formulæ $\int Vdx$ reſpondet æquatio pro curva hæc $N = o$, ita alteri formulæ $\int Vdy$ reſpondet æquatio $M = o$. Ex quo perſpicuum eſt coordinatas x & y inter ſe commutari poſſe.

Digitized by Google

SCHOLION I.

30. Apparet itaque in ſolutione hujuſmodi Problematum, quibus quæritur curva valorem formulæ $\int Z\,dx$ maximum minimumve habens, exiſtente Z functione ipſarum x, y, & p, perveniri ad æquationem differentialem ſecundi gradus, niſi in Z quantitas p unicam tantum habeat dimenſionem. Sæpe numero autem iſta æquatio differentialis ſecundi gradus integrationem admittit, de quo in ſingulis caſibus erit videndum. Interim hic annotaſſe juvabit, generaliter integrationem ſuccedere, ſi in functione Z omnino non inſit x, hoc eſt, ſi in ejus differentiali $dZ = M\,dx + N\,dy + P\,dp$ valor M evaneſcat, ita ut ſit tantum $dZ = N\,dy + P\,dp$. Cum enim pro curva inventa ſit hæc æquatio $N - \frac{dP}{dx} = 0$; multiplicetur ea per dy, & quia eſt $dy = p\,dx$, ea abibit in hanc $N\,dy - p\,dP = 0$, cui æquivalet iſta $N\,dy + P\,dp = P\,dp + p\,dP = dZ$, cujus integrale eſt $Z + C = Pp$, quæ æquatio jam tantum eſt differentialis primi gradus. Quoties ergo inter omnes curvas eidem abſciſſæ reſpondentes ea quæritur, in qua ſit valor formulæ $\int Z\,dx$ maximus vel minimus, atque Z tantum ſit functio ipſarum y & p, ita ut ſit $dZ = N\,dy + P\,dp$; tum, pro curva ſatisfaciente, ſtatim exhiberi poterit æquatio differentialis primi gradus iſta $Z + C = Pp$. Deinde vero etiam, ſi Z fuerit functio ipſarum x & p tantum, atque $dZ = M\,dx + P\,dp$, evaneſcente termino $N\,dy$, tum pro curva prodibit æquatio differentialis primi gradus. Nam, ob $dP = 0$, erit $P = C$, quæ pro curva quæſita dabit æquationem differentialem primi gradus tantum. Quod ſi autem inſuper M evaneſcat, ſeu Z functio ſit ipſius p tantum, & $dZ = P\,dp$; æquatio inventa $P = C$ tranſmutabitur in iſtam $P\,dp = C\,dp = dZ$, quæ denuo integrata dat $Z + D = Cp$. Hoc autem caſu, quia Z & P ſunt functiones ipſius p tantum, utraque æquatio $P = C$ & $Z + D = Cp$, præbebit pro p valorem conſtantem; ideoque æquationem hujus formæ $dy = n\,dx$, quæ indicat hujuſmodi Problematis ſatisfacere

cere

Digitized by Google

cere lineas rectas, & quidem quaſcunque utlibet ductas. Nam in æquatione $P = C$, cum C ſit conſtans arbitraria, valor ipſius p non ſolum conſtans, ſed etiam arbitrarius evadet; ex quo linea recta quæcunque reſultabit. Quamobrem ſi per data duo puncta curva duci debeat, in qua ſit $\int Z\,dx$ maximum vel minimum, ac Z ſit functio ipſius p tantum, tum ſatisfaciet linea recta per illa data duo puncta ducta.

SCHOLION II.

31. Quoniam ſupra jam vidimus in hujuſmodi Problematis coordinatas x & y inter ſe commutari, atque, ſi commodum videatur, applicatam y tanquam abſciſſam tractari poſſe, idem hoc quoque caſu confirmari juvabit. Sit igitur curva inveſtiganda in qua ſit $\int Z\,dy$ maximum vel minimum, exiſtente Z functione ipſarum x, y & p, & $dZ = M\,dx + N\,dy + P\,dp$. Hæc autem formula $\int Z\,dy$ ad noſtram formam reducta abit in $\int Zp\,dx$: in qua erit $d.\,Zp = Mp\,dx + Np\,dy + (Z + Pp)\,dp$: ex qua formulæ propoſitæ valor differentialis reſpondens erit $(Np\,dx - dZ - P\,dp - p\,dP)\,n\nu = (-M\,dx - 2P\,dp - p\,dP)\,n\nu$: & æquatio pro curva quæſita erit $0 = -M\,dx - 2P\,dp - p\,dP$; ſeu $0 = -M\,dy - d.\,Pp^2$. Quod ſi nunc ad ſimilitudinem oſtendendam, quia hic y tanquam abſciſſam conſideramus, ponamus $dx = \pi\,dy$, erit $p = \frac{1}{\pi}$ & $dp = -\frac{d\pi}{\pi\pi} = -pp\,d\pi$; erit $dZ = M\,dx + N\,dy - Ppp\,d\pi = M\,dx + N\,dy + \Pi\,d\pi$, ponendo $\Pi = -Ppp$; ut ſimilitudo terminorum conſervetur. Quapropter æquatio pro curva erit $0 = -M\,dy + d\Pi$; quæ eadem æquatio prodiiſſet, ſi in formula $\int Z\,dy$, applicata y in abſciſſam & viciſſim abſciſſa in applicatam tranſmutetur. Propoſita igitur quacunque formula indeterminata ex x & y horumque differentialibus compoſita, quæ debeat eſſe maxima vel minima; coordinatarum x & y utramlibet licebit tanquam abſciſſam tractare, ad eamque maximum minimumve accommodare.

EXEM-

Digitized by Google

Exemplum I.

32. *Inter omnes curvas ad eandem abſciſſam relatas, eam determinare, in qua ſit* $\int(Zdx + [Z]dy)$ *maximum vel minimum; exiſtentibus* Z *&* $[Z]$ *functionibus quibuſcunque ipſarum* x *&* y, *ita ut ſit* $dZ = Mdx + Ndy$ *&* $d[Z] = [M]dx + [N]dy$.

Ut formula hæc $\int(Zdx + [Z]dy)$ ad formam receptam reducatur, ponatur pdx loco dy; habebiturque hæc formula $\int(Z + [Z]p)dx$ maxima minimave efficienda. Differentietur ergo valor $Z + [Z]p$; eritque ejus differentiale $= + Mdx + Ndy + [M]pdx + [N]pdy + [Z]dp$.

Jam per regulam inventam, hinc pro curva quæſita iſta prodibit æquatio, $o = (N + [N]p)dx - d[Z] = (N + [N]p)dx - [M]dx - [N]dy$: quæ, ob $[N]pdx = [N]dy$, per dx diviſa dabit hanc æquationem pro curva quæſita algebraicam ſeu finitam $N - [M] = o$. ſeu $N = [M]$. Hinc intelligitur ſi formula propoſita $\int(Zdx + [Z]dy)$ fuerit determinata, ſeu differentiale $Zdx + [Z]dy$ ita comparatum, ut integrationem admittat; tum nullam lineam quæſito eſſe ſatisfacturam, ſeu potius omnes lineas æque ſatisfacere. Nam ſi $Zdx + [Z]dy$ integrationem admittit, per ſe erit $N = [M]$; uti alibi de formulis differentialibus duarum variabilium determinatis demonſtravimus; ideoque his caſibus prodit æquatio identica $o = o$. Hincque luculenter intelligitur, quod jam ante notavimus, maximi minimive formulam oportere eſſe formulam indeterminatam; alioquin enim omnes lineæ curvæ æque ſatisfacerent.

Exemplum II.

33. *Inter omnes lineas ad eandem abſciſſam relatas, determinare eam, cujus longitudo ſit minima; ſeu in qua ſit* $\int dx\sqrt{(1 + pp)}$ *minimum.*

Primum quidem apparet in hac quæſtione maximum non dari,

ri, cum linearum longitudo in infinitum augeri queat, manente abſciſſa eadem. Ita minimum tantum habebit locum, id quod ex ipſa Geometria elementari conſtat, in qua demonſtratur lineam rectam inter omnes alias lineas intra eoſdem terminos ſitas eſſe breviſſimam. Hoc igitur Exemplum ideo attuliſſe viſum eſt, cum ut conſenſus noſtræ Methodi cum veritate aliunde jam cognita intelligatur, tum etiam ut circumſtantia de duobus punctis arbitrariis, quæ ad hujus generis quæſtiones addi debet, melius percipiatur. Erit igitur, formula $\int dx\sqrt{(1+pp)}$ cum generali $\int Z\,dx$ comparata, $Z=\sqrt{(1+pp)}$, & $dZ=\frac{p\,dp}{\sqrt{(1+pp)}}$; unde fit $M=0$, $N=0$, & $P=\frac{p}{\sqrt{(1+pp)}}$. Quare, cum in genere æquatio pro linea quæſita ſit $N-\frac{dP}{dx}=0$, habebimus hoc caſu $dP=0$; ideoque $P=\frac{p}{\sqrt{(1+pp)}}=$ *Conſt.* ex qua æquatione oritur $p=$ *Conſt.* $=n$, ſeu $dy=n\,dx$, quæ denuo integrata dat $y=a+nx$. Non ſolum ergo patet lineam quæſitam eſſe rectam, ſed etiam, ob duas arbitrarias conſtantes a & n, rectam utcunque ductam. Quare ſi per data duo puncta linea duci jubeatur breviſſima, erit illa recta. Similiter autem intelligitur, ſi linea debeat inveniri, in qua ſit $\int Z\,dx$, ubi Z eſt functio ipſius p tantum, maximum vel minimum, tum lineam rectam tantum ſatisfacere; uti ante jam notavimus.

EXEMPLUM III.

34. *Inter omnes curvas ad eandem abſciſſam relatas, determinare eam, in qua ſit* $\int\frac{dx\sqrt{(1+pp)}}{\sqrt{x}}$ *maximum vel minimum.*

Hæc formula oritur, ſi quæratur linea celerrimi deſcenſus in hypotheſi gravitatis uniformis, ponendo axem in quo abſciſſæ capiuntur verticalem. Erit igitur $Z=\frac{\sqrt{(1+pp)}}{\sqrt{x}}$ & dZ

$= \frac{-dx\sqrt{(1+pp)}}{2x\sqrt{x}} + \frac{p\,dp}{\sqrt{x(1+pp)}}$; unde fit $M = \frac{-\sqrt{(1+pp)}}{2x\sqrt{x}}$; $N = 0$, & $P = \frac{p}{\sqrt{x(1+pp)}}$. Cum jam curva quæſita exprimatur æquatione $N - \frac{dP}{dx} = 0$; erit $dP = 0$, & P ($= \frac{p}{\sqrt{x(1+pp)}}$) $=$ *Conſt.* $= \frac{1}{\sqrt{a}}$; quæ reducta præbet $ap^2 = x + p^2x$, & $p = \frac{dy}{dx} = \sqrt{\frac{x}{a-x}}$, ſeu $y = \int dx \sqrt{\frac{x}{a-x}}$, quæ æquatio indicat, curvam quæſitam eſſe Cycloidem ſuper baſi horizontali natam, & cuſpidem in ſuprema axis regione habentem: quæ adeo per data duo quæcunque puncta duci poterit.

EXEMPLUM IV.

35. *Inter omnes curvas eidem abſciſſæ reſpondentes, eam determinare in qua ſit* $\int y^n\, dx \sqrt{(1+pp)}$ *maximum vel minimum.*

Pro hac ergo formula propoſita erit $Z = y^n \sqrt{(1+pp)}$ & $dZ = ny^{n-1}\,dy\sqrt{(1+pp)} + \frac{y^n p\,dp}{\sqrt{(1+pp)}}$; ita ut fiat $M = 0$, & $N = ny^{n-1}\sqrt{(1+pp)}$ atque $P = \frac{y^n p}{\sqrt{(1+pp)}}$.

Quoniam igitur eſt $M = 0$; ſtatim pro curva quæſita habebitur iſta æquatio ſemel jam integrata $Z + C = Pp$ (30), quæ noſtro caſu fit $y^n \sqrt{(1+pp)} + ma^n = \frac{y^n pp}{\sqrt{(1+pp)}}$. Quod ſi ponatur conſtans $a = 0$, prodibit $1 + pp = pp$, ſeu $p = \infty$, ſatisfacietque linea recta normalis ad axem. Generatim vero lineæ ſatisfacientes reperientur ex æquatione, quæ abit in $y^n + ma^n \sqrt{(1+pp)} = 0$, ſeu $y^{2n} = m^2a^{2n} + m^2a^{2n}p^2$; quæ dat p ($= \frac{dy}{dx}$) $= \frac{\sqrt{(y^{2n} - m^2a^{2n})}}{ma^n}$, & $x = \int \frac{ma^n\,dy}{\sqrt{(y^{2n} - m^2a^{2n})}}$;

quæ

Digitized by Google

quæ linea per data duo puncta duci poteſt. Si fuerit $n = -\frac{1}{2}$, ita ut $\int \frac{dx\sqrt{(1+pp)}}{\sqrt{y}}$ debeat eſſe maximum vel minimum; pariter prodire debet linea brachyſtochrona ad axem horizontalem relata; eritque pro ea $x = \int dy \sqrt{\frac{y}{a-y}}$; quæ cum præcedente omnino congruit, dummodo coordinatæ x & y inter ſe commutentur. Erit ſcilicet, ut ante, curva ſatisfaciens Cyclois ſuper baſi horizontali rotando generata, qualem per data duo quæcunque puncta ducere licet.

EXEMPLUM V.

36. *Inter omnes curvas eidem abſciſſæ reſpondentes eam determinare, in qua ſit* $\frac{\int y dy^3}{dx^2+dy^2}$ *maximum vel minimum*

Formula hæc ad formam conſuetam, ope ſubſtitutionis $dy = pdx$, reducta, abit in hanc $\int \frac{yp^3 dx}{1+pp}$; eaque reperiri ſolet, ſi quæratur ſolidum rotundum rotatione curvæ circa axem ortum, quod ſecundum axis directionem in fluido motum minimam patiatur reſiſtentiam: reſiſtentia namque hoc caſu proportionalis cenſetur formulæ $\int \frac{y dy^3}{dx^2+dy^2}$ ſeu $\int \frac{yp^3 dx}{1+pp}$. Erit ergo $Z = \frac{yp^3}{1+pp}$, & $dZ = \frac{p^3 dy}{1+pp} + \frac{y dp(3pp+p^4)}{(1+pp)^2}$; ita ut fiat $M = 0$, $N = \frac{p^3}{1+pp}$ & $P = \frac{p^2 y(3+pp)}{(1+pp)^2}$. Cum igitur ſit $M = 0$; una integratio generaliter ſuccedit, eritque æquatio pro curva quæſita $Z + C = Pp$, ſeu $\frac{yp^3}{1+pp} + a = \frac{p^3 y(3+pp)}{(1+pp)^2}$; quæ abit in hanc $a(1+pp)^2 = 2p^3 y$. Hujus æquationis autem evolutio non ita poteſt inſtitui ut eliminetur p; quare conveniet utramque coordinatam y & x per eandem variabilem p definiri. Ac primo quidem eſt $y = \frac{a(1+pp)^2}{2p^3}$. Deinde, ob $dy = pdx$;

 erit

Digitized by Google

erit $dx = \frac{dy}{p}$ & $x = \int \frac{dy}{p} = \frac{y}{p} + \int \frac{y\,dp}{pp}$. Quod ſi ergo loco y valor inventus ſubſtituatur, prodibit $x = \frac{a(1+pp)^2}{2p^4} + a\int \frac{dp(1+pp)^2}{2p^3} = \frac{a}{2}\left(\frac{3}{4p^4} + \frac{1}{pp} + 1 + lp\right)$: ex quibus curvæ conſtructio poterit confici, logarithmis in ſubſidium vocandis.

EXEMPLUM VI.

37. *Invenire curvam, in qua iſta formula* $\int yx\,dx\sqrt{(1+pp)}$ *ſit maximum minimumve.*

Erit ergo $Z = yx\sqrt{(1+pp)}$, atque $dZ = y\,dx\sqrt{(1+pp)} + x\,dy\sqrt{(1+pp)} + \frac{yxp\,dp}{\sqrt{(1+pp)}}$. Hanc ob rem habebitur $M = y\sqrt{(1+pp)}$, $N = x\sqrt{(1+pp)}$ & $P = \frac{yxp}{\sqrt{(1+pp)}}$; unde æquatio pro curva formabitur hæc $N\,dx = dP$, quæ ſuggerit $x\,dx\sqrt{(1+pp)} = \frac{p^2 x\,dx + yp\,dx}{\sqrt{(1+pp)}} + \frac{yx\,dp}{(1+pp)^{3:2}}$, ſeu $x\,dx - y\,dy = \frac{yx\,dp}{1+pp}$, ob $dy = p\,dx$. Hæc eſt æquatio differentialis ſecundi gradus, & quanquam, ope idonearum ſubſtitutionum, ea ad formam ſimpliciter differentialem reduci poteſt, eo quod variabiles x & y ubique eundem dimenſionum numerum conſtituunt; tamen æquatio iſta differentialis ita eſt comparata, ut neque integrari neque ſeparari poſſit; deduci ſcilicet poteſt ad æquationem hujus formæ $\frac{du}{u^3} + \frac{dv}{v^3} = \frac{v\,dv(1+u^2)}{u^3}$. Quod cum ita ſit, neque æquatio inventa $x\,dx - y\,dy = \frac{yx\,dp}{1+pp}$ ad formam vel ſimpliorem vel commodiorem revocari poteſt; hincque nihil admodum de natura curvæ inventæ judicare licet. Interim tamen illa æquatio potentiâ duas arbitrarias conſtantes involvit, ex quo curva ſatisfaciens per bina puncta data duci poteſt.

EXEM-

Digitized by Google

EXEMPLUM VII.

38. *Invenire curvam, in qua ſit* $\int (xx+yy)^n dx\sqrt{(1+pp)}$ *maximum vel minimum.*

Cum hîc ſit $Z = (xx+yy)^n\sqrt{(1+pp)}$, erit $dZ = 2n(xx+yy)^{n-1}(xdx+ydy)\sqrt{(1+pp)} + \frac{(xx+yy)^n pdp}{\sqrt{(1+pp)}}$, ergo $N = 2n(xx+yy)^{n-1}y\sqrt{(1+pp)}$ & $P = \frac{(xx+yy)^n p}{\sqrt{(1+pp)}}$; ex quo pro curva quæſita iſta habebitur æquatio $2n(xx+yy)^{n-1}ydx\sqrt{(1+pp)} = d\frac{(xx+yy)^n p}{\sqrt{(1+pp)}} = \frac{2n(xx+yy)^{n-1}p(xdx+ydy)}{\sqrt{(1+pp)}} + \frac{dp(xx+yy)^n}{(1+pp)^{3:2}}$, quæ per $(xx+yy)^{n-1}$ diviſa, ac per $\sqrt{(1+pp)}$ multiplicata, abit in $2nydx = 2nxdy + \frac{(xx+yy)dp}{1+pp}$ ſeu $\frac{2n(ydx-xdy)}{xx+yy} = \frac{dp}{1+pp}$. Hujus æquationis utrumque membrum integrabile eſt per quadraturam circuli, fitque integrale $2n\,A\,\text{tang.}\frac{x}{y} = A\,\text{tang.}\,p + A\,\text{tang.}\,k = A\,\text{tang.}\frac{p+k}{1-pk}$: unde fiet $\frac{x}{y} = \text{tang.}\frac{1}{2n}A\,\text{tang.}\frac{k+p}{1-kp} = T$; eritque T functio algebraïca ipſius p, dummodo ſit $2n$ numerus rationalis. Cum ergo ſit $x = Ty$, ſeu $y = \frac{x}{T}$, erit $dy = pdx = \frac{dx}{T} - \frac{xdT}{TT}$, ſive $xdT = Tdx - pTTdx$; ideoque $\frac{dx}{x} = \frac{dT}{T-pTT} + \frac{Tdp}{1-pT} - \frac{Tdp}{1-pT}$; unde prodit $lx = l\frac{T}{1-pT} - \int\frac{Tdp}{1-pT}$, quæ quidem ad conſtruendam curvam abunde ſatisfaciunt. Verum ut harum curvarum, quæ pro definitis exponentis n valoribus prodeunt, natura melius cognoſcatur, Caſus nonnullos contemplabimur.

I. Sit $n = \frac{1}{2}$, & $2n = 1$; erit $A\text{tang.}\frac{x}{y} = A\,\text{tang.}\frac{k+p}{1-kp}$,

Digitized by Google

ideoque $\frac{x}{y} = \frac{k + p}{1 - kp} = \frac{kdx + dy}{dx - kdy}$, ſeu $xdx - kxdy = kydx + ydy$; quæ integrata præbet $x^2 - y^2 = 2kxy + C$; quæ eſt æquatio pro Hyperbola æquilatera.

II. Sit $n = 1$, & $2n = 2$; erit 2 A tang. $\frac{x}{y}$ = A tang. $\frac{k + p}{1 - kp}$, ſeu A tang. $\frac{2xy}{yy - xx}$ = A tang. $\frac{k + p}{1 - kp}$; unde fit $\frac{2xy}{yy - xx} = \frac{kdx + dy}{dx - kdy}$, ſeu $2xydx - 2kxydy = kyydx - kxxdx + yydy - xxdy$; quæ integrata dat $yx^2 = ky^2x - \frac{1}{3}kx^3 + \frac{1}{3}y^3 + C$, ſive $y^3 + 3ky^2x - 3yx^2 - kx^3 = C$.

III. Sit $n = \frac{3}{2}$, ſeu $2n = 3$; erit 3 A tang. $\frac{x}{y}$ = A tang. $\frac{3y^2x - x^3}{y^3 - 3yx^2}$ = A tang. $\frac{kdx + dy}{dx - kdy}$; hincque $3y^2xdx - 3ky^2xdy - x^3dx + kx^3dy = ky^3dx + y^3dy - 3kyx^2dx - 3yx^2dy$; quæ integrata dat $\frac{3}{2}y^2x^2 - ky^3x - \frac{1}{4}x^4 + kyx^3 - \frac{1}{4}y^4 = C$, ſeu $y^4 + 4ky^3x - 6y^2x^2 - 4kyx^3 + x^4 = C$.

Ex his jam caſibus colligi poterit æquatio integralis pro valore quocunque ipſius n. Cum enim ſit $2n$ A tang. $\frac{x}{y}$ = A tang.

$$\frac{2ny^{2n-1}x - \frac{2n(2n-1)(2n-2)}{1.\ 2.\ 3.}y^{2n-3}x^3 + \&c.}{y^{2n} - \frac{2n(2n-1)}{1.\ 2.}y^{2n-2}x^2 + \frac{2n(2n-1)(2n-2)(2n-3)}{1.\ 2.\ 3.\ 4.}y^{2n-4}x^4 - \&c.}$$

$$= \frac{(y + x\sqrt{-1})^{2n} - (y - x\sqrt{-1})^{2n}}{(y + x\sqrt{-1})^{2n}\sqrt{-1} + (y - x\sqrt{-1})^{2n}\sqrt{-1}};$$ fiet $\frac{kdx + dy}{dx - kdy}$

$$= \frac{(y + x\sqrt{-1})^{2n} - (y - x\sqrt{-1})^{2n}}{(y + x\sqrt{-1})^{2n}\sqrt{-1} + (y - x\sqrt{-1})^{2n}\sqrt{-1}};$$ quæ reducta præbet

$$kdx(y + x\sqrt{-1})^{2n}\sqrt{-1} + kdx(y - x\sqrt{-1})^{2n}\sqrt{-1} + kdy(y + x\sqrt{-1})^{2n} - kdy(y - x\sqrt{-1})^{2n}$$
$$= dy(y + x\sqrt{-1})^{2n}\sqrt{-1} + dy(y - x\sqrt{-1})^{2n}\sqrt{-1} - dx(y + x\sqrt{-1})^{2n} + dx(y - x\sqrt{-1})^{2n}$$ cujus integrale eſt

eſt $k(y+x\sqrt{-1})^{2n+1} - k(y-x\sqrt{-1})^{2n+1} = -\frac{1}{\sqrt{-1}}(y+x\sqrt{-1})^{2n+1} - \frac{1}{\sqrt{-1}}(y-x\sqrt{-1})^{2n+1} + C$, ſeu $C = (y+x\sqrt{-1})^{2n+1}(k\sqrt{-1}+1) + (y-x\sqrt{-1})^{2n+1}(1-k\sqrt{-1})$. At eſt generaliter $(y+x\sqrt{-1})^{2n+1} + (y-x\sqrt{-1})^{2n+1} = 2(yy+xx)^{(2n+1):2}$ coſ. $2n$ A tang. $\frac{x}{y}$, atque . . . . .
$\frac{(y+x\sqrt{-1})^{2n+1} - (y-x\sqrt{-1})^{2n+1}}{\sqrt{-1}} = 2(yy+xx)^{(2n+1):2}$ ſin. $2n$ A tang. $\frac{x}{y}$. Quibus valoribus ſubſtitutis, prodibit æquatio integralis ab imaginariis libera hæc $2k(yy+xx)^{(2n+1):2}$ ſin. $2n$ A tang. $\frac{x}{y} = 2(yy+xx)^{(2n+1):2}$ coſ. $2n$ A tang. $\frac{x}{y} + C$: vel, ob conſtantes arbitrarias k & C, iſta $C = (yy+xx)^{(2n+1):2}(k$ ſin. $2n$ A tang. $\frac{x}{y} + h$ coſ. $2n$ A tang. $\frac{x}{y})$, quæ æquatio ſemper eſt algebraïca, dummodo fuerit n numerus rationalis. Vel ſi arcus quidam circularis arbitrarius ponatur $= g$, curva quæſita hujuſmodi æquatione $C = (yy+xx)^{(2n+1):2}$ ſin $(g + 2n$ A tang. $\frac{x}{y})$ exprimi poteſt; poſito radio circuli, quem hic contemplamur, $= 1$.

SCHOLION III.

39. Si ergo, inter omnes curvas eidem abſciſſæ reſpondentes, ea debeat inveniri, in qua ſit $\int Z dx$ maximum vel minimum, exiſtente Z functione ipſarum x, y & p, ita ut ſit $dZ = M dx + N dy + P dp$; pro curva quæſita iſta habebitur æquatio

$N-$

$N - \frac{dP}{dx} = 0$. Quoniam autem in Problemate præcedente annotavimus, ſi Z tantum fuerit functio ipſarum x & y, tum Methodo vulgari ſolutionem abſolvi poſſe: nam ut $\int Z dx$ ſit maximum minimumve, etiam $Z dx$, ac proinde Z tale eſſe oportet, reſpectu ad x habito; & hanc ob rem differentiale ipſius dZ, ſumto x conſtante, nihilo æquale poſitum dabit æquationem pro curva quæſita. Similis Methodus ſuccederet in præſente Problemate, ſi modo in differentiali ipſius Z, quod oritur poſito x conſtante, atque eſt $N dy + P dp$, relatio inter differentialia dy & dp pateret, ut per dy diviſio inſtitui, atque valor finitus nihilo æquandus erui poſſet. Cum autem iſtam relationem inter dy & dp, ſine qua Methodus maximorum & minimorum vulgaris adhiberi nequit, a priori definire etiamnum non liceat, poterimus eam a poſteriori aſſignare: Quia enim inventa eſt æquatio pro curva quæſita hæc $N - \frac{dP}{dx} = 0$; intelligitur, hanc ex illa $N dy + P dp$, ſeu $N + \frac{P dp}{dy}$ oriri potuiſſe, ſi conſtitiſſet eſſe $-\frac{dP}{dx} = \frac{P dp}{dy}$, ſeu $0 = dP + \frac{P dp}{p}$; ob $dy = p dx$. Quocirca relatio illa inter differentialia dy & dp ita erit comparata, ut contineatur æquatione $p dP + P dp = 0$; quæ proprietas ad hanc redit ut conſiderari debeat Pp tanquam conſtans. Hinc ad Problemata reſolvenda, in quibus curva quæritur habens valorem formulæ $\int Z dx$ maximum vel minimum, exiſtente $dZ = M dx + N dy + P dp$; valor ipſius Z debet differentiari, atque in differentiali $M dx + N dy + P dp$, loco $M dx$ poni debeat o, $N dy$ immutatum relinqui, tum vero loco $P dp$ ſcribi $- p dP$; & id quod emergit nihilo æquale poni. Hoc enim pacto obtinebitur $N dy - p dP = 0$; quæ æquatio, ob $dy = p dx$, tranſit in hanc $N - \frac{dP}{dx} = 0$, quæ eſt ea ipſa quam invenimus. Deſideratur itaque Methodus a reſolutione geometrica & lineari libera, qua pateat in tali inveſtigatione maximi minimive loco $P dp$ ſcribi debere $- p dP$.

PRO-

Propositio IV. Problema.

40. *Si* Z *fuerit functio ipſarum* x, y, p *&* q, *ita ut ſit* $dZ = Mdx + Ndy + Pdp + Qdq$; *invenire, inter omnes curvas eidem abſciſſæ reſpondentes, eam in qua ſit* $\int Z\,dx$ *maximum vel minimum.*

S O L U T I O.

Valor formulæ integralis $\int Z\,dx$ evolvitur in binas has ſeries $Z\,dx + Z'dx + Z''dx + Z'''dx +$ &c. & $Z_{,}dx + Z_{,,}dx + Z_{,,,}dx$ + &c. quarum aggregatum maximum erit vel minimum, ſi ſingulorum terminorum valores differentiales, qui oriuntur augendo applicatam y' particula $n\nu$, colligantur, & nihilo æquentur. Tali autem applicatæ y' incremento mutationem patiuntur litteræ y'; p, p'; $q_{,}, q, q'$; adeoque ii tantum termini in quibus iſtæ litteræ inſunt, hoc eſt termini $Z_{,}dx$, $Z\,dx$ & $Z'dx$. Ad horum terminorum augmenta, ex tranſlatione puncti n in ν orta, invenienda, differentientur ii, eritque

$$d.\,Z'dx = dx\,(M'dx + N'dy' + P'dp' + Q'dq')$$
$$d.\,Z\,dx = dx\,(M\,dx + N\,dy + P\,dp + Q\,dq)$$
$$d.\,Z_{,}dx = dx\,(M_{,}dx + N_{,}dy_{,} + P_{,}dp_{,} + Q_{,}dq_{,})$$

Jam vero, quia abſciſſa x ab illa tranſlatione non afficitur, ponendum eſt ubique $dx = 0$: deinde vero reliquorum differentialium valores ex tranſlatione puncti n in ν orti, per primam hujus Capitis Propoſitionem ita ſe habebunt:

$dy' = +\,n\nu$	$dp' = -\dfrac{n\nu}{dx}$	$dq' = +\dfrac{n\nu}{dx^2}$
$dy = 0$	$dp = +\dfrac{n\nu}{dx}$	$dq = -\dfrac{2n\nu}{dx^2}$
$dy_{,} = 0$	$dp_{,} = 0$	$dq_{,} = +\dfrac{n\nu}{dx^2}$

His differentialium per n, expreſſorum valoribus ſubſtitutis, prodibit ſequens valor differentialis, $n\nu.\,dx\,(N - \frac{P'}{dx} + \frac{P}{dx} + \frac{Q'}{dx^2} - \frac{2Q}{dx^2} + \frac{Q_{,}}{dx^2}) = n\nu.\,dx\,(N - \frac{dP}{dx} + \frac{ddQ_{,}}{dx^2}) = n\nu.\,dx\,(N - \frac{dP}{dx} + \frac{ddQ_{,}}{dx^2})$ ob $ddQ_{,} = ddQ$. Quamobrem pro curva quæſita iſta habebitur æquatio $N - \frac{dP}{dx} + \frac{ddQ}{dx^2} = 0$. *Q. E. I.*

COROLL. I.

41. Quod ſi ergo in maximi minimive formula $\int Z dx$ inſint etiam differentialia ſecundi gradus, ſeu, quod idem eſt, ſi Z fuerit functio ipſarum x, y, p & q; ita ut ſit $dZ = Mdx + Ndy + Pdp + Qdq$; æquatio pro curva quæſita erit $N - \frac{dP}{dx} + \frac{ddQ}{dx^2} = 0$; quæ facile ex differentiali ipſius Z formabitur.

COROLL. II.

42. Si quantitas Q ipſa involvit q vel differentio-differentiale ipſius y; tum ddQ continebit differentialia quarti ordinis, in hocque genere erit æquatio pro curva inventa. Ex quo curva ſatisfaciens per quatuor data puncta traduci poterit.

COROLL. III.

43. Si igitur in Q contineatur q, tum Problema ita determinate proponendum erit, ut inter omnes curvas per quatuor data puncta ductas ea definiatur, in qua $\int Z dx$ ſit maximum vel minimum.

SCHOLION I.

44. Ponamus in Q non contineri q, ut inveſtigemus cujuſnam

nam gradus futura ſit æquatio differentialis reſultans. Accidit autem hoc, ſi maximi minimive formula propoſita fuerit hujuſmodi $\int Zq\,dx$, exiſtente Z functione tantum ipſarum x, y & p: ita ut ſit $dZ = M dx + N dy + P dp$. Hinc igitur erit $d.Zq = Mq\,dx + Nq\,dy + Pq\,dp + Z\,dq$: unde pro curva quæſita orietur æquatio hæc $o = Nq - \frac{P\,dq + q\,dP}{dx}$ $+ \frac{dM\,dx + dN\,dy + N\,ddy + P\,ddp + dP\,dp}{dx^2}$, ſeu $o = 2Nq$ $+ \frac{dM + p\,dN}{dx}$, vel $o = 2N\,dp + dM + p\,dN$: quæ æquipollet tantum æquationi differentialis ſecundi gradus, propter $dp = \frac{ddy}{dx}$ quod ineſt. Si igitur curva deſideretur, in qua ſit $\int Zq\,dx$ maximum vel minimum, exiſtente Z functione ipſarum x, y & p, atque $dZ = M dx + N dy + P dp$; pro curva quæſita habebitur æquatio $o = dM + 2N\,dp + p\,dN$.

COROLL. IV.

45. Ut revertamur ad æquationem inventam $N - \frac{dP}{dx} + \frac{ddQ}{dx^2}$ $= o$; patet eam fore generaliter integrabilem ſi ſit $N = o$, hoc eſt ſi in Z non contineatur y; prodibit enim integrando $C - P + \frac{dQ}{dx} = o$. Si inſuper ſit $P = o$, altera integratio ſuccedit, qua prodit $Cx + D - Q = o$.

COROLL. V.

46. Si ſit $M = o$, pariter una integratio in genere ſuccedit: cum enim ſit $dZ = N dy + P dp + Q dq$; multiplicetur æquatio $N - \frac{dP}{dx} + \frac{ddQ}{dx^2} = o$ per dy, ſeu $p dx$, habebitur $N dy$ $- p\,dP + \frac{p\,ddQ}{dx} = o$. Addatur $dZ - N dy - P dp -$

Digitized by Google

$Qdq = 0$, orietur $dZ - pdP - Pdp + \frac{pddQ}{dx} - Qdq = 0$; cujus integrale eſt $Z - Pp + \frac{pdQ}{dx} - Qq = C$.

COROLL. VI.

47. Si fuerit & $M = 0$ & $N = 0$; erit primo, ob $N = 0$, ut ſupra, $C - P + \frac{dQ}{dx} = 0$. Deinde cum ſit $dZ = Pdp + Qdq$, multiplicetur illa æquatio per dp, ſeu qdx, erit $Cdp - Pdp + qdQ = 0$: addatur $Pdp + Qdq - dZ = 0$; prodibit $Cdp + Qdq + qdQ - dZ = 0$, cujus integralis eſt $Cp + D + Qq - Z = 0$.

SCHOLION II.

48. Si nexum æquationis inventæ pro curva quæſita, quæ habeat $\int Z dx$ maximum minimumve, cum differentiali ipſius Z inſpiciamus; determinare licebit relationem inter differentialia dy, dp & dq, ut differentiale ipſius Z nihilo æquale poſitum, præbeat æquationem pro curva quæſita. Cum enim ſit $dZ = Mdx + Ndy + Pdp + Qdq$; comparetur cum hac forma æquatio pro curva, $N - \frac{dP}{dx} + \frac{ddQ}{dx^2} = 0$, ſeu hæc per $dy = pdx$ multiplicata, quæ erit $Ndy - pdP + \frac{pddQ}{dx} = 0$; unde patet, in differentiali ipſius Z, loco Mdx ſcribi debere 0, at terminum Ndy invariatum relinqui, porro loco Pdp ſcribendum eſſe $-pdP$, ac loco Qdq poni debere $\frac{pddQ}{dx}$. Verum quoad hæc a priori pateant, præſtabit formam æquationis inventæ retinere, quippe quæ facile memoria teneri poteſt. Cæterum notandum eſt Problemata huc pertinentia omnino eſſe nova, neque adhuc ab iis qui alias de hoc argumento ſcripſerunt pertractata. Alias enim Scriptores maximi minimive formulas contemplari non conſueverunt, niſi in quibus ad ſummum differen-

ferentialia coordinatarum primi gradus ineſſent. Quamobrem eo magis erit operæ pretium naturam hujuſmodi Problematum accuratius indagare, atque inprimis oſtendere, quomodo curvæ ſatisfacientes quatuor puncta, per quæ tranſeant, ad ſui determinationem admittant. Hunc in finem ſequentia Exempla adjicere viſum eſt; atque in ſingulis indicare, quæ ad majorem illuſtrationem facere poterunt.

EXEMPLUM I.

49. *Invenire curvam, in qua ſit* $\int \frac{y^n\, ddy}{x^m\, dy}$ *maximum vel minimum.*

Iſta maximi minimive formula, ope ſubſtitutionum $dy = p\,dx$; & $ddy = q\,dx^2$, abit in hanc $\int \frac{y^n q\,dx}{x^m p}$; quæ cum ſit ſimilis formulæ §. 44 tractatæ $\int Zq\,dx$, ubi in Z tantum x, y & p contineri poſuimus, fiet, comparatione inſtituta, $Z = \frac{y^n}{x^m p}$, & $dZ = -\frac{my^n\,dx}{x^{m+1}p} + \frac{ny^{n-1}dy}{x^m p} - \frac{y^n\,dp}{x^m p^2}$; unde erit $M = -\frac{my^n}{x^{m+1}p}$, & $N = \frac{ny^{n-1}}{x^m p}$; hincque $Np = \frac{ny^{n-1}}{x^m}$.

Cum igitur pro curva quæſita, inventa ſit hæc æquatio $0 = dM + 2\,N dp + p\,dN = dM + N dp + d.Np$, habebimus pro noſtro caſu hanc æquationem $0 = \frac{m(m+1)y^n\,dx}{x^{m+2}p} - \frac{mny^{n-1}dy}{x^{m+1}p} + \frac{my^n\,dp}{x^{m+1}pp} + \frac{ny^{n-1}dp}{x^m p} + \frac{n(n-1)\,y^{n-2}\,dy}{x^m} -$

$-\frac{mny^{n-1}dx}{x^{m+1}}$: quæ multiplicata per $\frac{x^{m+1}p^2}{y^{n-1}}$ mutatur in hanc $0 = m(m+1)ydy - mnxpdy + mxydp + nx^2pdp + \frac{n(n-1)x^2p^2dy}{y} - mnxpdy$, ſeu $0 = m(m+1)y^2dy - 2mnxypdy + n(n-1)x^2p^2dy + mxy^2dp + nx^2ypdp$: quæ eſt æquatio differentialis ſecundi gradus, quæ, poſito $y = e^{\int vdx}$ reducetur ad iſtam primi gradus $m(m+1)vdx + mxdv - m(2n-1)xv^2dx + nx^2vdv + n^2x^2v^3dx = 0$. Quod ſi autem ponamus $m = 0$, ita ut maximum minimumve eſſe debeat $\int\frac{y^n ddy}{dy}$; habebitur hæc æquatio $(n-1)pdy + ydp = 0$, quæ integrata dabit $y^{n-1}p = C$, ſeu $y^{n-1}dy = Cdx$; hæcque denuo integrata præbet $y^n = Cx + D$. Sin autem ponamus $n = 0$, ita ut maximum minimumve eſſe debeat hæc formula $\int\frac{ddy}{x^m dy}$; erit $(m+1)dy + xdp = 0$, ſeu $(m+1)pdx + xdp = 0$, cujus integrale eſt $x^{m+1}p = C$, ſeu $dy = Cx^{-m-1}dx$; quæ denuo integrata dat $y = \frac{C}{x^m} + D$. Patet autem in his curvis inventis formulam propoſitam fieri maximum, non vero minimum; nam ſi ſumatur linea recta, ob $ddy = 0$, manifeſtum eſt valorem formulæ propoſitæ minorem fore pro recta linea quam pro curvis inventis.

SCHOLION III.

50. Ratio hîc aſſignari poteſt, cur hujuſmodi quæſtiones, in quibus $\int Zqdx$ maximum minimumve eſſe debet, deducant tantum ad æquationem differentialem ſecundi gradus, ideoque quæſtionibus præcedentis Problematis potius ſint adnumerandæ, ſiquidem Z fuerit functio ipſarum x & y & p. Nam per reductio-

ductionem integralium formula $\int Zq\,dx$, ſeu $\int \frac{Z\,ddy}{dx}$, reduci poteſt ad talem formam $Y + \int V\,dx$, in qua Y & V ſint functiones ipſarum x, y & p tantum, non amplius involventes q. Cum igitur Y ſit quantitas abſoluta, atque idcirco in maximi minimive inquiſitionem non cadat, formula $\int Zq\,dx$ fiet maxima vel minima, ſi $\int V\,dx$ talis reddatur; adeo ut hujuſmodi formulæ $\int Zq\,dx$ reduci queant ad præcedentis Problematis ſtatum; unde mirum non eſt, quod pro curvis ſatisfacientibus æquatio differentialis ſecundi gradus duntaxat reperiatur. Quo autem memorata reductio formulæ $\int Zq\,dx$ ſeu $\int Z\,dp$ ad $Y + \int V\,dx$ melius percipiatur; ponamus, cum Y ſit functio ipſarum x, y & p, eſſe $dY = \varrho\,dx + \sigma\,dy + \tau\,dp = (\varrho + \sigma p)\,dx + \tau\,dp$; & ex æqualitate $\int Z\,dp = Y + \int V\,dx$, erit $Z\,dp = (\varrho + \sigma p)\,dx + \tau\,dp + V\,dx$; unde concluditur $\tau = Z$ & $V = -\varrho - \sigma p$. Quamobrem ipſa hæc reductio ſequenti modo inſtituetur; integretur formula $Z\,dp$ poſitis x & y conſtantibus, & integrale erit functio ipſarum x, y & p, quæ vocetur Y. Deinde differentietur hæc functio Y, ponendo p conſtans, & differentiale negative ſumtum dabit $V\,dx$, eritque V functio ipſarum x, y & p non continens q. Quoties igitur reddi debet hujuſmodi formula $\int Zq\,dx$ maximum minimumve; ac Z eſt functio ipſarum x & y & p; tum quæſtio, etiamſi videatur ad præſens Problema pertinere, tamen ſtatim ad Problema præcedens reducetur. Ita ſi ſumamus formulam $\int \frac{y^n\,ddy}{dy}$, ſeu $\int \frac{y^n\,dp}{p}$; hæc facile transformatur in $y^n\,lp - n\int y^{n-1}\,dy\,lp$: unde maximum vel minimum eſſe debebit hæc formula $\int y^{n-1}\,dy\,lp$, ſeu $\int y^{n-1}\,p\,dx\,lp$, quæ per præcedens Problema tractata, dabit $Z = y^{n-1}\,p\,lp$, & $dZ = (n-1)y^{n-2}\,dy\,p\,lp + y^{n-1}\,dp(1 + lp)$; eritque $M = 0$, $N = (n-1)y^{n-2}\,p\,lp$ & $P = y^{n-1}(1 + lp)$. At ob $M = 0$, ſupra §. 30 pro curva quæſita inventa eſt hæc æquatio $Z + C = Pp$, quæ ad noſtrum caſum accommodata præbet

fiet $y^{n-1}plp + C = y^{n-1}p + y^{n-1}plp$, ſive $y^{n-1}p = C$; quæ eſt ea ipſa æquatio, quam ante pro eodem caſu in ſolutione Exempli invenimus. Hanc ob rem ad Exempla huic Problemati propria progrediamur.

EXEMPLUM II.

Fig. 5. 51. *Invenire curvam* AM, *quæ cum ſua evoluta* AR *& radio oſculi* mR *in quovis loco applicato, minimum ſpatium* ARm *includat.*

Poſitis abſciſſa $AM = x$, applicata $Mm = y$; erit radius oſculi $mR = -\frac{(1+pp)^{3:2}}{q}$; area autem ARm eſt $= \int \frac{1}{2} mR.\, dx\sqrt{(1+pp)}$; ex qua minimum eſſe oportet hanc formulam $\int \frac{(1+pp)^2\, dx}{q}$. Erit itaque $Z = \frac{(1+pp)^2}{q}$, & $dZ = \frac{4(1+pp)p\,dp}{q} - \frac{(1+pp)^2\, dq}{qq}$; unde fit $M = 0$, $N = 0$, $P = \frac{4(1+pp)p}{q}$, & $Q = -\frac{(1+pp)^2}{qq}$. Cum nunc ſit $M = 0$ & $Z = 0$; erit, per Coroll. 6, æquatio pro curva quæſita $Z = D + Cp + Qq$, ſeu $\frac{(1+pp)^2}{q} = D + Cp - \frac{(1+pp)^2}{q}$, hoc eſt $2(1+pp)^2 = Dq + Cpq$. Quoniam porro eſt $dp = q\,dx$, ſeu $q = \frac{dp}{dx}$, erit $2dx = \frac{(D+Cp)\,dp}{(1+pp)^2}$, cujus integrale eſt $x = \frac{a}{1+pp} + 2b\int \frac{dp}{(1+pp)^2} = \frac{a+bp}{1+pp} + b\int \frac{dp}{1+pp} + c$: mutatis pro lubitu conſtantibus, habebitur $x = \frac{a+bp+cpp}{1+pp} + b$ A tang. p. Deinde quia eſt $dy = p\,dx$, erit $y = \int p\,dx = px - \int x\,dp$; ideoque $y = \frac{ap+bp^2+cp^3}{1+pp} + bp$ A tang. $p - \int \frac{(a+bp+cpp)\,dp}{1+pp} - b\int dp$ A tang. p

A tang. $p = \frac{ap + bp^2 + cp^3}{1 + pp} - \int \frac{(a + cpp)\,dp}{1 + pp}$, ob $b\int dp$ A tang. $p = bp$ A tang. $p - b\int \frac{p\,dp}{1 + pp}$. Hinc erit $y = f + \frac{ap + bp^2 + cp^3}{1 + pp}$ $+ (c - a)$ A tang. $p - cp = \frac{f + (a - c)p + (b + f)pp}{1 + pp}$ $+ (c - a)$ A tang. p. Atque ex his quidem ipſarum x & y valoribus per p inventis, curva quæſita per data quatuor puncta duci atque conſtrui poteſt. Verum ut ipſa curva qualis ſit cognoſcatur, eliminetur A tang. p; eritque A tang. $p = \frac{x}{b} - \frac{\frac{a}{b} - p - \frac{c}{b}pp}{1 + pp}$ $= \frac{y}{c - a} - \frac{\frac{f}{c - a} + p - \frac{(b + f)}{c - a}pp}{1 + pp}$; atque hinc $(c - a)x - by$ $= \frac{(ac - aa - bf) + 2b(c - a)p + (cc - ac - bb - bf)pp}{1 + pp}$.

Quoniam autem ipſa curva non mutatur, etiamſi coordinatæ conſtante quantitate vel augeantur vel diminuantur, erit $(c - a)x - by = \frac{bb - (c - a)^2 + 2b(c - a)p}{1 + pp}$; poſitoque a loco $c - a$, habebitur $ax - by = \frac{bb - aa + 2abp}{1 + pp}$; & ſubtracta conſtante bb, erit $ax - by = \frac{-aa + 2abp - bbpp}{1 + pp}$ hincque $\sqrt{(by - ax)} = \frac{bp - a}{\sqrt{(1 + pp)}}$. Ponatur arcus curvæ $= w$; erit $dw = dx\sqrt{(1 + pp)}$; unde emerget iſta æquatio $dw = \frac{b\,dy - a\,dx}{\sqrt{(by - ax)}}$; atque porro $w = 2\sqrt{(by - ax)}$. Exprimit autem $by - ax$ multiplum abſciſſæ ſuper alio quodam axe fixo aſſumtæ, cui adeo quadratum arcus reſpondentis eſt proportionale. Ex quo intelligitur curvam quæſito reſpondentem eſſe Cycloïdem, quæ per quatuor data puncta determinatur, atque ſic deſcripta inter omnes alias curvas per eadem quatuor puncta ductas, minimum cum ſua evoluta concludit ſpatium.

Concluſio hæc ideo aliquantum difficilior facta eſt, quod Cycloïs pro recta quacunque inſtar axis aſſumpta quæſito ſatisfaciat, atque æquatio pro axe quocunque admodum fiat intricata. Si autem vel a vel b poſuiſſemus $= 0$, quo quidem extenſio ſolutionis non fuiſſet reſtricta; æquatio pro Cycloïde ſtatim prodiiſſet.

EXEMPLUM III.

52. *Invenire curvam, in qua ſit* $\int \varrho^n\, dw$, *denotante* ϱ *radium oſculi, &* dw *elementum curvæ, maximum vel minimum.*

Per poſitiones ante factas eſt $dw = dx\sqrt{(1+pp)}$ & $\varrho = \frac{(1+pp)^{3:2}}{q}$; unde maximi minimive formula erit $\int \frac{(1+pp)^{(3n+1):2}}{q^n}dx$; hincque fit $Z = \frac{(1+pp)^{(3n+1):2}}{q^n}$ & $dZ =$

$$\frac{(3n+1)(1+pp)^{(3n-1):2}\,p\,dp}{q^n} - \frac{n(1+pp)^{(3n+1):2}\,dq}{q^{n+1}}.$$

Quamobrem erit $M = 0$, $N = 0$, $P = \frac{(3n+1)(1+pp)^{(3n-1):2}\,p}{q^n}$, & $Q = \frac{-n(1+pp)^{(3n+1):2}}{q^{n+1}}$. Cum autem ſit $M = 0$ & $N = 0$, erit, per §. 47, $= Cp + D + Qq$; ideoque $\frac{(1+pp)^{(3n+1):2}}{q^n} = Cp + D - \frac{n(1+pp)^{(3n+1):2}}{q^n}$, ſeu $(n+1)(1+pp)^{(3n+1):2} = (Cp+D)\,q^n$; atque hinc $q = \frac{(1+pp)^{(3n+1):2n}}{\sqrt[n]{(C+Dp)}} = \frac{dp}{dx}$; ergo $dx =$

$dp\sqrt[n]{}$

$dp\sqrt[n]{\frac{C+Dp}{(1+pp)^{(3n+1):2}}}$, atque $dy = p\,dp\sqrt[n]{\frac{C+Dp}{(1+pp)^{(3n+1):2}}}$. Hic autem merito ſuſpicari licet, æquationem futuram eſſe ſimpliciorem, ſi alius axis accipiatur. Hanc ob rem concipiamus alium axem in quo abſciſſa ſit $= t$, applicata $= v$; ſitque $dv = s\,dt$; ac ponatur $x = \frac{\alpha t + \beta v}{\gamma}$ & $y = \frac{\beta t - \alpha v}{\gamma}$, poſito $\gamma = \sqrt{(\alpha^2 + \beta^2)}$. Erit ergo $dx = \frac{\alpha dt + \beta s dt}{\gamma}$ & $dy = \frac{\beta dt - \alpha s dt}{\gamma}$ atque $\frac{dy}{dx} = p = \frac{\beta - \alpha s}{\alpha + \beta s}$, & $(1+pp) = \frac{\gamma^2(1+ss)}{(\alpha+\beta s)^2}$, & $dp = -\frac{\gamma\gamma ds}{(\alpha+\beta s)^2}$. Porro autem erit $C + Dp = \frac{\alpha C + \beta D + s(\beta C - \alpha D)}{\alpha + [b]s}$, & $(1+pp)^{(3n+1):3n} = \frac{\gamma^{(3n+1):n}(1+ss)^{(3n+1):2n}}{(\alpha+\beta s)^{(3n+1):n}}$. His ſubſtitutis, erit $dx = \frac{\alpha dt + \beta dv}{\gamma} = \frac{\alpha(\alpha+\beta s)ds}{(1+ss)^{(3n+1):4n}}$, poſito $\beta C = \alpha D$, & mutata conſtante. Porro autem fit $dy = \frac{\beta dt - \alpha dv}{\gamma} = \frac{a(\beta - \alpha s)ds}{(1+ss)^{(3n+1):2n}}$, & conjunctim prodit $dt = \frac{a\,ds}{(1+ss)^{(3n+1):2n}}$, & $dv = \frac{a s\,ds}{(1+ss)^{(3n+1):2n}}$. Cum nunc has coordinatas æque x & y appellare poſſimus ac præcedentes, fiet $s = p$, atque $dx = \frac{a\,dp}{(1+pp)^{(3n+1):2n}}$, & $dy = \frac{ap\,dp}{(1+pp)^{(3n+1):2n}}$, quæ ex præcedentibus oriuntur, ſi ibi ponatur $D = 0$: ex quo perſpicuum eſt, latitudini ſolutionis ſuperioris, in qua inerat $C + Dp$, nihil omnino decedere, etſi ponatur $D = 0$. Eadem ſcilicet prodit linea curva, quicunque

 valor

valor litteræ D tribuatur, etiamſi alia æquatio inter x & y proveniat; verumtamen ad alium axem relata. Notare interim convenit pluribus caſibus curvam algebraïcam ſatisfacere; quorum quaſi primus eſt ſi $n = \frac{1}{2}$, quo erit $x = \int \frac{adp}{(1+pp)^{5:2}} = \frac{ap(1+\frac{2}{3}pp)}{(1+pp)^{3:2}}$, & $y = \int \frac{apdp}{(1+pp)^{5:2}} = - \frac{\frac{1}{3}a}{(1+pp)^{3:2}}$; unde fiet $(1+pp)^{3:2} = -\frac{a}{3y}$ & $pp = \sqrt[3]{\frac{aa}{9yy}} - 1$, quibus ſubſtitutis reſultat $x = (2\sqrt[3]{\frac{aay}{9}} + y)\sqrt{(\sqrt[3]{\frac{aa}{9yy}} - 1)}$, æquatio algebraïca pro curva, caſu quo eſt $n = \frac{1}{2}$.

EXEMPLUM IV.

53. *Invenire curvam, in qua ſit valor hujus formulæ* $\int \frac{ydydx^2}{ddy}$ *omnium minimus.*

Patet primo maximum locum habere non poſſe, quia in linea recta fit $ddy = 0$; ideoque valor formulæ propoſitæ infinite magnus. Quamobrem videndum eſt in quanam linea curva fiat valor formulæ $\int \frac{ydydx^2}{ddy}$ minimus. Hæc autem formula per ſubſtitutiones noſtras abit in hanc $\int \frac{ypdx}{q}$; eritque $Z = \frac{yp}{q}$, & $dZ = \frac{pdy}{q} + \frac{ydp}{q} - \frac{ypdq}{qq}$; erit ergo $M = 0$, $N = \frac{p}{q}$, & $P = \frac{y}{q}$, & $Q = -\frac{yp}{qq}$. Quoniam autem eſt $M = 0$; curva quæſita ſequenti exprimetur æquatione $Z - Pp - Qq + \frac{pdQ}{dx} = C$, ut Coroll. 5 eſt oſtenſum. Quamobrem iſta proveniet æquatio $\frac{yp}{q} - \frac{p}{dx} d. \frac{yp}{qq} = C$, ſeu $\frac{ydy}{pq} + \frac{adx}{p} = \frac{pdy}{qq} + \frac{ydp}{qq} - \frac{2ypdq}{q^3}$, ob $dy = pdx$. Quia vero eſt $dp = qdx$, erit

erit $\frac{y\,dp}{qq} = \frac{y\,dx}{q} = \frac{y\,dy}{pq}$; ideoque $\frac{\alpha\,dx}{p} = \frac{p\,dy}{qq} - \frac{2yp\,dq}{q^3}$, vel $\frac{\alpha\,dp}{pp} = \frac{dy}{q} - \frac{2y\,dq}{qq}$. Si ponatur conſtans $\alpha = 0$, hæc æquatio fiet integrabilis, eritque $y = bqq$ & $q = \sqrt{\frac{y}{b}} = \frac{dp}{dx} = \frac{p\,dp}{dy}$, unde fit $p\,dp = dy\,\sqrt{\frac{y}{b}}$; atque integrando $\frac{pp}{2} = \frac{2}{3}y\,\sqrt{\frac{y}{b}} + c$, ſeu, mutatis conſtantibus, $pp = \frac{y\sqrt{y} + a\sqrt{a}}{b\sqrt{b}}$, & $p = \sqrt{\frac{y^{3:2} + a^{3:2}}{b^{3:2}}} = \frac{dy}{dx}$; hincque $dx = dy\sqrt{\frac{b^{3:2}}{y^{3:2} + a^{3:2}}}$. Ponatur denuo $a = 0$, erit $x = \frac{b\sqrt{c}}{\sqrt{y}}$ & $xxy = b^2c$; quæ eſt æquatio maxime ſpecialis pro curva quæſtioni ſatisfaciente.

EXEMPLUM V.

54. *Invenire curvam, in qua ſit valor hujus formulæ* $\int q^n dx$, *ſeu* $\int \frac{ddy^n}{dx^{2n-1}}$, *maximus vel minimus*.

Habetur ergo $Z = q^n$, & $dZ = nq^{n-1}\,dq$; unde erit $M = 0$, $N = 0$, $P = 0$, & $Q = nq^{n-1}$. Cum igitur æquatio pro curva ſatisfaciente ſit hæc $\frac{ddQ}{dx^2} = 0$, erit $dQ = \alpha\,dx$ & $Q = q^{n-1} = \alpha x + \beta$; hincque $q = (\alpha x + \beta)^{1:(n-1)} = \frac{dp}{dx}$; ex quo fiet $p = (\alpha x + \beta)^{n:(n-1)} + \gamma = \frac{dy}{dx}$, & tandem $y = (\alpha x + \beta)^{(2n-1):(n-1)} + \gamma x + \delta$; ubi coefficientes per integrationes ingreſſos in conſtantibus ſumus complexi. Curvæ igitur ſatisfacientes perpetuo ſunt algebraïcæ; excepto caſu, quo eſt $n = \frac{1}{2}$, tum enim poſtrema integra-

Digitized by Google

tio præbebit $y = \frac{1}{\alpha} l(\alpha x + \beta) + \gamma x + \delta$. Quod ad caſum $n = 1$ attinet, ille in inveſtigationem maximorum & minimorum nequidem incurrit; cum formula $\int q\,dx$ non ſit indeterminata, ſed determinatum valorem, puta p, ob $q dx = dp$, referat. Cæterum patet, evaneſcente termino $(\alpha x + \beta)^{(2n-1):(n-1)}$, lineam rectam quæſito ſatisfacere, ob $y = \gamma x + \delta$. Scilicet ſi quatuor puncta data, per quæ curva quæſita tranſire debeat, ſint in directum poſita; tum ipſa linea recta, præ omnibus reliquis lineis per eadem quatuor puncta tranſeuntibus, quæſito ſatisfaciet.

Exemplum VI.

55. *Invenire curvam, in qua ſit* $\int \frac{xp\,dx}{yq}$ *maximum vel minimum.*

Quia eſt $Z = \frac{xp}{yq}$, erit $dZ = \frac{p\,dx}{yq} - \frac{xp\,dy}{y^2 q} + \frac{x\,dp}{yq} - \frac{xp\,dq}{yqq}$; ideoque $M = \frac{p}{yq}$; $N = -\frac{xp}{y^2 q}$; $P = \frac{x}{yq}$, & $Q = \frac{-xp}{yq^2}$. Quorum terminorum cum nullus evaneſcat, æquatio pro curva quæſita erit $\frac{-xp}{y^2 q} - \frac{1}{dx} d. \frac{x}{yq} - \frac{1}{dx^2} d^2. \frac{xp}{yq^2} = 0$, ſeu $0 = \frac{xp\,dx^2}{y^2 q} + \frac{dx^2}{yq} - \frac{x\,dx\,dy}{y^2 q} - \frac{x\,dx\,dq}{yq^2} + d.\left(\frac{p\,dx}{yq^2} - \frac{xp\,dy}{y^2 q^2} + \frac{x\,dp}{yq^2} - \frac{2xp\,dq}{yq^3}\right)$; vel $0 = q^2 dx^2 (3yq - 2p^2)(y - xp) - 4yq\,dx\,dq\,(xyq - xp^2 + yp) + 6xy^2 p\,dq^2 - 2xy^2 pq\,ddq$. Quæ eſt æquatio differentialis quarti ordinis, quæ utrum integrari poſſit, an non, haud facile patet; neque etiam operæ pretium eſt in modum eam integrandi diligentius inquirere; quoniam hic caſus non ex ſolutione Problematis alicujus utilis eſt natus,

Digitized by Google

natus, ſed fortuito excogitatus. Hoc autem Exemplum ideo adjicere viſum eſt, ut caſus habeatur, quo ſolutio non ſolum ad æquationem differentialem quarti ordinis aſcendit, ſed etiam neque per ſubſidia generalia ſupra allata ad gradum inferiorem perduci queat. Præcedentia enim Exempla cuncta ita ſunt comparata, ut per regulas generales in Corollariis expoſitas ſtatim æquatio pro curva quæſita inferioris gradus differentialis erui potuerit.

PROPOSITIO V. PROBLEMA.

56. *Invenire curvam, in qua ſit valor formulæ* $\int Z\,dx$ *maximus vel minimus, exiſtente* Z *ejuſmodi functione, quæ differentialia cujuſvis gradus involvat, ita ut ſit* $dZ = M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + S\,ds + T\,dt + \&c.$

SOLUTIO.

Quoniam translatio puncti n in ν præcedentia elementa magis afficit, quam ſequentia; unicum enim ſequens elementum afficit, at in præcedentia eo ulterius extenditur, quo altiorum ordinum differentialia adſint; hanc ob rem expediet aliquam anteriorem applicatam, uti Hh, pro prima accipere, ita ut mutatio ex particula nν applicatæ Nn adjecta non citra Hh porrigatur; id quod eveniet ſi in Z differentialia non ultra ſextum ordinem aſcendant. Sufficiet autem valorem ipſius dZ ad terminum $T\,dt$ uſque extendere, quia ex ipſa ſolutione modus facile colligetur eam ad quotcunque ulteriores terminos accommodandi. Præterea, quia in hoc Problemate præcedentia omnia continentur, conſtabit ſimul ſolutionem perpetuo eandem prodire, quæcunque applicata particula quadam infinite parva, uti nν, augeatur. Sit igitur $AH = x$, & $Hh = y$, reſpondebunt ſingulis punctis abſciſſæ H, I, K, L, M, N, O &c. valores litterarum p, q, r, s, t &c. ut ſequitur: Fig. 4.

H

Digitized by Google

H	$y,$	$p,$	$q,$	$r,$	$s,$	t
I	$y',$	$p',$	$q',$	$r',$	$s',$	t'
K	$y'',$	$p'',$	$q'',$	$r'',$	$s'',$	t''
L	$y''',$	$p''',$	$q''',$	$r''',$	s'''	t'''
M	$y^{IV},$	$p^{IV},$	q^{IV}	$r^{IV},$	$s^{IV},$	t^{IV}
N	$y^{V},$	$p^{V},$	$q^{V},$	$r^{V},$	$s,$	$t^{V}.$

Hi autem ſinguli valores a tranſlatione n in ν ſequentia augmenta accipient, quæ ex Propoſitione prima, debita mutatione ſignorum adhibita, ita ſe habebunt.

$dy = 0$	$dy' = 0$	$dy'' = 0$	$dy''' = 0$	$dy^{IV} = 0$	$dy^{V} = + n\nu$
$dp = 0$	$dp' = 0$	$dp'' = 0$	$dp''' = 0$	$dp^{IV} = + \frac{n\nu}{dx}$	$dp^{V} = - \frac{n\nu}{dx}$
$dq = 0$	$dq' = 0$	$dq'' = 0$	$dq''' = + \frac{n\nu}{dx^2}$	$dq^{IV} = - \frac{2n\nu}{dx^2}$	$dq^{V} = + \frac{n\nu}{dx^2}$
$dr = 0$	$dr' = 0$	$dr'' = + \frac{n\nu}{dx^3}$	$dr^{IV} = - \frac{3n\nu}{dx^3}$	$dr^{IV} = + \frac{3n\nu}{dx^3}$	$dr^{V} = - \frac{n\nu}{dx^3}$
$ds = 0$	$ds' = + \frac{n}{dx}$	$ds'' = - \frac{4n\nu}{dx^4}$	$ds''' = + \frac{6n\nu}{dx^4}$	$ds^{IV} = \frac{4n\nu}{dx^4}$	$ds^{V} = + \frac{n\nu}{dx^4}$
$dt = + \frac{n\nu}{dx^5}$	$dt' = - \frac{5n}{dx^5}$	$dt'' = + \frac{10n\nu}{dx^5}$	$dt''' = - \frac{10n\nu}{dx^5}$	$dt^{IV} = + \frac{5n\nu}{dx^5}$	$dt^{V} = - \frac{n\nu}{dx^5}$

Quoniam porro valor formulæ $\int Z\,dx$ abſciſſæ A H reſpondet, iſque a tranſlatione puncti n in ν non mutatur, ſequentibus abſciſſæ elementis valores formulæ $\int Z\,dx$ reſpondent, qui in hac Tabula exhibentur.

Elemento	reſpondet
HI	$Z\,dx$
IK	$Z'\,dx$
KL	$Z''\,dx$
LM	$Z'''\,dx$
MN	$Z^{IV}\,dx$
NO	$Z^{V}\,dx$

Adho-

Digitized by Google

Ad horum valorum incrementa, ex tranſlatione puncti n in ν oriunda, invenienda, ſingulos hos valores differentiari, locoque differentialium dy, dp, dq, dr, ds, dt cum ipſorum derivativis valores ſupra aſſignatos & per nν expreſſos ſubſtitui oportet; eritque ut ſequitur:

$$d.Zdx = \mathrm{n}\nu.dx\left(\frac{T}{dx^5}\right)$$

$$d.Z'dx = \mathrm{n}\nu.dx\left(\frac{S'}{dx^4} - \frac{5T'}{dx^5}\right)$$

$$d.Z''dx = \mathrm{n}\nu.dx\left(\frac{R''}{dx^3} - \frac{4S''}{dx^4} + \frac{10T''}{dx^5}\right)$$

$$d.Z'''dx = \mathrm{n}\nu.dx\left(\frac{Q'''}{dx^2} - \frac{3R'''}{dx^3} + \frac{6S'''}{dx^4} - \frac{10T'''}{dx^5}\right)$$

$$d.Z^{IV}dx = \mathrm{n}\nu.dx\left(\frac{P^{IV}}{dx} - \frac{2Q^{IV}}{dx^2} + \frac{3R^{IV}}{dx^3} - \frac{4S^{IV}}{dx^4} + \frac{5T^{IV}}{dx^5}\right)$$

$$d.Z^{V}dx = \mathrm{n}\nu.dx\left(N^{V} - \frac{P^{V}}{dx} + \frac{Q^{V}}{dx^2} - \frac{R^{V}}{dx^3} + \frac{S^{V}}{dx^4} - \frac{T^{V}}{dx^5}\right)$$

Quia jam hæc ſola elementa a tranſpoſitione puncti n in ν alterantur & incrementa capiunt, ſumma horum incrementorum dabit integrum valorem differentialem, quem formula $\int Zdx$ ad totam abſciſſam AZ extenſa accipit; qui igitur erit

$$\mathrm{n}\nu.dx\left\{\begin{array}{l} +N^{V} \\ -\dfrac{P^{V}+P^{IV}}{dx} \\ +\dfrac{Q^{V}-2Q^{IV}+Q'''}{dx^2} \\ -\dfrac{R^{V}+3R^{IV}-3R'''+R''}{dx^3} \\ +\dfrac{S^{V}-4S^{IV}+6S'''-4S''+S'}{dx^4} \\ -\dfrac{T^{V}+5T^{IV}-10T'''+10T''-5T'+T}{dx^5} \end{array}\right.$$

Singula autem hæc membra per differentialia commode & ſuccincte exprimi poterunt: erit enim,

$$-P^{V}+P^{IV}=-dP^{IV}$$
$$+Q^{V}-2Q^{IV}+Q'''=+ddQ'''$$
$$-R^{V}+3R^{IV}-3R'''+R''=-d^3R''$$
$$+S^{V}-4S^{IV}+6S'''-4S''+S'=+d^4S'$$
$$-T^{V}+5T^{IV}-10T'''+10T''-5T'+T=-d^5T$$

Quamobrem formulæ $\int Z\,dx$ integer valor differentialis ex particula n ν ortus, erit $= n\nu.\,dx\,(N^{V}-\frac{dP^{IV}}{dx}+\frac{ddQ'''}{dx^2}-\frac{d^3R''}{dx^3}+\frac{d^4S'}{dx^4}-\frac{d^5T}{dx^5})$. Hic autem, quia omnes termini ſunt homogenei, ſignaturæ tuto omitti poſſunt, evaneſcit enim diſcrimen inter N^{V} & N, itemque inter dP^{IV} & dP, reliquaque. Quocirca habebitur formulæ $\int Z\,dx$ iſte valor differentialis

$$n\nu.\,dx\,(N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\frac{d^4S}{dx^4}-\frac{d^5T}{dx^5}):$$

ex quo ſimul valor differentialis formulæ $\int Z\,dx$ colligi poteſt; ſi in Z altiora etiam differentialia ineſſent. Quare ſi curva quæratur, quæ habeat $\int Z\,dx$ maximum vel minimum pro data abſciſſa, fueritque $dZ = M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + S\,ds + T\,dt +$ &c. erit primum formulæ $\int Z\,dx$ valor differentialis hic:

$$n\nu.\,dx\,(N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\frac{d^4S}{dx^4}-\frac{d^5T}{dx^5}+\text{\&c.})$$

Hincque pro curva quæſita orietur iſta æquatio

$$0 = N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\frac{d^4S}{dx^4}-\frac{d^5T}{dx^5}+\text{\&c.}\quad Q.E.I.$$

COROLL. I.

57. In formula $\int Z\,dx$, uti eam tractavimus, quantitas Z continet differentialia quinti gradus: ſi quidem in differentiali ipſius

Digitized by Google

ipſius $dZ = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + Tdt$ terminus Tdt eſt ultimus. Cum igitur in T adhuc inſint differentialia quinti gradus ſeu t, perſpicuum eſt æquationem pro curva quæſita fore differentialem decimi gradus.

Coroll. II.

58. Hinc intelligitur perpetuo æquationem differentialem pro curva ad gradum duplo altiorem aſcendere debere, quam fuerit ipſa formula maximi minimive. Ponimus enim, in ultimo termino Tdt, quantitatem T adhuc t in ſe complecti: niſi enim hoc eſſet, duobus gradibus æquatio deprimeretur, uti ex §. 50 colligere licet.

Coroll. III.

59. Si igitur in Z differentialia gradus n contineantur, tum æquatio pro curva differentialis erit gradus $2n$: & hanc ob rem totidem novas conſtantes arbitrarias poteſtate in ſe continet.

Coroll. IV.

60. Ob tot igitur conſtantes arbitrarias, totidem puncta ad Problema determinandum propoſita eſſe oportet; ita ſcilicet Problema, ut ſit determinatum, enuntiari debebit; Inter omnes curvas per data $2n$ puncta tranſeuntes, determinare eam in qua ſit $\int Zdx$ maximum vel minimum, ſiquidem quantitas Z complectatur differentialia n gradus.

Coroll. V.

61. Ob n igitur numerum integrum, numerus punctorum, quo Problema determinabitur, ſemper erit par. Sic, vel nullum punctum, vel duo, vel quatuor, vel ſex, vel octo puncta, & ita porro, ad Problematis determinationem requiruntur.

Digitized by Google

SCHOLION I.

62. Ex gradu *differentialitatis* igitur, ad quem æquatio pro curva inventa aſſurgit, vel ex numero punctorum per quæ curvam ſatisfacientem tranſire oportet, hujuſmodi Problemata commode in Claſſes diſtribui poterunt. Ad primam igitur Claſſem ea referentur Problemata, in quibus abſolute quæritur linea curva, quæ pro data abſciſſa habeat valorem $\int Z\,dx$ maximum vel minimum; talia Problemata cum in Propoſitione ſecunda continentur, tum etiam in tertia, iis caſibus quos §. §. 26 & 27 expoſuimus; his ſcilicet caſibus ſolutio præbet curvam determinatam quæſito ſatisfacientem. Claſſis ſecunda ea complectitur Problemata, quorum ſolutio ad æquationem differentialem ſecundi gradus perducit: hæcque duo puncta ad ſui determinationem poſcunt; & ita proponi debent, ut inter omnes curvas per data duo puncta tranſeuntes ea definiatur, in qua ſit $\int Z\,dx$ maximum vel minimum: cujuſmodi Problemata in Propoſitione tertia ſoluta dedimus. Porro ad tertiam Claſſem pertinent Problemata in Propoſitione quarta tractata, quæ ita ſe habent; ut inter omnes curvas per quatuor data puncta tranſeuntes determinetur ea quæ habeat $\int Z\,dx$ maximum vel minimum. Simili modo quarta Claſſis poſtulat ad determinationem ſex puncta, quinta octo, & ita porro, quas Claſſes omnes in præſente Problemate ſumus complexi. Cæterum etſi æquatio inventa ad tantum differentialium gradum aſcendit, tamen ſæpius generaliter integrationem unam vel plures admittit, cujuſmodi caſus in præcedentibus Problematibus nonnullos exhibuimus. Hanc ob rem videamus etiam quibus caſibus æquatio noſtra generalis integrationem, vel unam vel plures, admittat; ut in allatis Exemplis ſtatim videre liceat, utrum ea in his caſibus contineantur an ſecus. Hujuſmodi autem caſus potiſſimum ſunt duo, in quorum altero eſt $N = 0$, in altero $M = 0$, a quibus deinceps alii caſus pendent, quos hic evolvemus.

Digitized by Google

Casus I.

63. Sit in maximi minimive formula $\int Z\,dx$ terminus $N=0$; ita ut ſit $dZ = M\,dx + P\,dp + Q\,dq + R\,dr + S\,ds + \&c.$

Hoc ergo caſu æquatio pro curva erit hæc, $0 = -\frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \frac{d^5T}{dx^5} + \&c.$ quæ, per dx multiplicata, fit integrabilis, prodibitque

$$0 = A - P + \frac{dQ}{dx} - \frac{ddR}{dx^2} + \frac{d^3S}{dx^3} - \frac{d^4T}{dx^4} + \&c.$$

Casus II.

64. Sit & $N=0$ & $P=0$, ita ut ſit $dZ = M\,dx + Q\,dq + R\,dr + S\,ds + T\,dt + \&c.$

Quoniam eſt $N=0$, una integratio jam ſucceſſit, habeturque pro curva quæſita iſta æquatio modo inventa, poſito etiam $P=0$:

$$0 = A + \frac{dQ}{dx} - \frac{ddR}{dx^2} + \frac{d^3S}{dx^3} - \frac{d^4T}{dx^4} + \&c.$$

quæ per dx multiplicata denuo integrari poterit, eritque

$$0 = Ax - B + Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \frac{d^3T}{dx^3} + \&c.$$

Casus III.

65. Si fuerit & $N=0$ & $P=0$ & $Q=0$, ita ut ſit $dZ = M\,dx + R\,dr + S\,ds + T\,dt + \&c.$

Bini valores N & P evaneſcentes jam præbuerunt hanc æquationem bis integratam $0 = Ax - B + Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \frac{d^3T}{dx^3} + \&c.$ in qua ſi ponatur $Q=0$, & multiplicetur per dx, obtinebitur ſequens æquatio ter integrata:

$$0 = Ax^2 - Bx + C - R + \frac{dS}{dx} - \frac{ddT}{dx^2} + \&c.$$

 Ex

Digitized by Google

Ex quo jam apparet, ſi inſuper fuerit $R = 0$, tum etiam quartam integrationem locum habere, & ita porro.

Casus IV.

66. Si fuerit $M = 0$, ita ut ſit $dZ = N\,dy + P\,dp + Q\,dq + R\,dr + S\,ds +$ &c.

Æquatio pro curva quæſita ante prodiit $0 = N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} -$ &c. quæ multiplicetur per $dy = p\,dx$, & tum addatur $dZ - N\,dy - P\,dp - Q\,dq - R\,dr - S\,ds -$ &c. prodibit.

$$0 = dZ - p\,dP + \frac{p\,ddQ}{dx} - \frac{p\,d^3R}{dx^2} + \frac{p\,d^4S}{dx^3} - \text{\&c.}$$
$$- P\,dp - Q\,dq - R\,dr - S\,ds - \text{\&c.}$$

cujus integrale aſſignari poteſt; erit enim

$$\begin{aligned} 0 = A + Z - Pp &+ \frac{p\,dQ}{dx} - \frac{p\,ddR}{dx^2} + \frac{p\,d^3S}{dx^3}\ \text{\&c.} \\ &- Qq + \frac{q\,dR}{dx} - \frac{q\,ddS}{dx^2} \\ &- Rr + \frac{r\,dS}{dx}. \\ &- Ss \end{aligned}$$

vel $0 = A + Z - Pp + \frac{p\,dQ - Q\,dp}{dx} - \frac{p\,ddR - dp\,dR + R\,ddp}{dx^2} + \frac{p\,d^3S - dp\,ddS + dS\,ddp - S\,d^3p}{dx^3} -$ &c. cujus termini, quomodo ulterius progrediantur, ſi in dZ inſint ſequentia differentialia $T\,dt$, $U\,du$ &c. ſponte patet.

Casus V.

67. Si ſit & $M = 0$, & $N = 0$; ita ut ſit $dZ = P\,dp + Q\,dq + R\,dr + S\,ds +$ &c.

Quia eſt $N = 0$, una integratio per caſum primum inſtituatur,

Digitized by Google

tur, habebiturque $o = A - P + \frac{dQ}{dx} - \frac{ddR}{dx^2} + \frac{d^3S}{dx^3} -$&c. quæ multiplicetur per $dp = qdx$, ad eamque addatur $o = -dZ + Pdp + Qdq + Rdr + Sds +$&c. quo facto prodibit iſta æquatio integrabilis

$$o = Adp - dZ + qdQ - \frac{qddR}{dx} + \frac{qd^3S}{dx^2} - \text{\&c.}$$
$$+ Qdq + Rdr + Sds + \text{\&c.}$$

cujus integrale erit

$$o = Ap - B - Z + Qq - \frac{qdR}{dx} + \frac{qddS}{dx^2} \text{\&c.}$$
$$+ Rr - \frac{rdS}{dx}$$
$$+ Ss$$

ſeu $o = Ap - B - Z + Qq - \frac{qdR - Rdq}{dx} + \frac{qddS - dqdS + Sddq}{dx^2} - \frac{qd^3T - dqddT + dTddq - Td^3q}{dx^3}$ &c.

Casus VI.

68. Sit & $M = o$ & $N = o$ & $P = o$, ita ut ſit $dZ = Qdq + Rdr + Sds + Tdt +$ &c.

Ob $N = o$ & $P = o$, per Caſum ſecundum, duæ integrationes locum habent, eritque æquatio, pro curva quæſita, hæc,

$$o = Ax - B + Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \frac{d^3T}{dx^3} + \text{\&c.}$$

ad quam per $dq = rdx$ multiplicatam ſi addatur $o = dZ - Qdq - Rdr - Sds - Tdt -$ &c. habebitur iſta æquatio denuo integrabilis:

$$o = Axdq - Bdq + dZ - rdR + \frac{rddS}{dx} - \frac{rd^3T}{dx^2} + \text{\&c.}$$
$$- Rdr - Sds - Tdt - \text{\&c.}$$

cujus integrale eſt

$o =$

Digitized by Google

$$0 = Axq - Bq + C + Z - Rr + \frac{rdS}{dx} - \frac{rddT}{dx^2} \quad \&c.$$
$$- Ap \qquad - Ss + \frac{sdT}{dx}$$
$$- Tt$$

ſeu $0 = A(xq - p) - Bq + C + Z - Rr + \frac{rdS - Sdr}{dx}$
$- \frac{rddT - drdT + Tddr}{dx^2} + \&c.$

CASUS VII.

69. Si fuerit $M = 0$, $N = 0$, $P = 0$, & $Q = 0$, ita ut ſit $dZ = Rdr + Sds + Tdt + \&c.$

Ob $N = 0$, & $Q = 0$, Caſus tertius iſtam ſuppeditat æquationem pro curva jam ter integratam,

$$0 = Ax^2 - Bx + C - R + \frac{dS}{dx} - \frac{ddT}{dx^2} + \&c.$$

ad quam per $dr = sdx$ multiplicatam addatur $0 = -dZ + Rdr + Sds + Tdt + \&c.$ quo facto prodibit iſta æquatio,

$$0 = Ax^2dr - Bxdr + Cdr - dZ + sdS - \frac{sddT}{dx} + \&c.$$
$$+ Sds + Tdt + \&c.$$

quæ integrata dabit hanc,

$$0 = Ax^2r - Bxr + Cr - D - Z + Ss - \frac{sdT}{dx} + \&c.$$
$$- 2Axq + Bq \qquad + Tt$$
$$+ 2Ap$$

ſeu $0 = A(x^2r - 2xq + 2p) - B(xr - q) + Cr - D - Z + Ss - \frac{sdT - Tds}{dx} + \&c.$

SCHO-

SCHOLION II.

70. Horum caſuum ope, quorum numerum ulterius augere liceret, ſi commodum videretur, ſæpe-numero Problemata admodum expedite reſolvi poterunt. Quod ſi enim Problema quodpiam contineatur in aliquo iſtorum Caſuum, qui unam pluresve integrationes per ſe admittat, ſtatim formari poterit æquatio pro curva, ſemel vel aliquoties jam integrata, quæ propterea ulterius facilius tractari poterit. Quod ut diſtinctius pateat, ſimulque uſus hujus poſtremi Problematis, quo in maximi minimive formula $\int Z\,dx$ differentialia ſecundum gradum ſuperantia inſunt, declaretur, unicum Exemplum afferre juvabit.

EXEMPLUM.

71. ***Inter omnes curvas eidem abſciſſæ reſpondentes, definire eam, cujus evoluta, cum ſua ipſius evoluta, intra radios evolutæ maximum minimumve ſpatium complectatur.***

Poſitis, pro curva quæſita, abſciſſa $= x$ & applicata $= y$; ſit elementum curvæ $= dw$, & ejus radius oſculi $= \varrho$; erit elementum ipſius evolutæ $= d\varrho$, & hujus radius oſculi $= \frac{\varrho\, d\varrho}{dw}$: unde area comprehenſa inter evolutam curvæ quæſitæ, ipſiuſque evolutam, erit $= \frac{1}{2}\int \frac{\varrho\, d\varrho^2}{dw}$; quæ ergo expreſſio maxima minimave eſt reddenda. Cum autem ſit $dw = dx\sqrt{(1+pp)}$, & $\varrho = \frac{(1+pp)^{\frac{3}{2}}}{q}$; erit $d\varrho = 3(1+pp)^{\frac{1}{2}}\, p\,dx - \frac{(1+pp)^{\frac{3}{2}}\, r\,dx}{qq}$, & $d\varrho^2 = (1+pp)\,dx^2\left(9pp - \frac{6(1+pp)r}{qq} + \frac{(1+pp)^2 rr}{q^4}\right)$ atque $\frac{\varrho}{dw} = \frac{1+pp}{q\,dx}$. Maximi minimive formula itaque eſt $\int \frac{(1+pp)^2}{q}\,dx\,(9pp -$

$$\frac{6(1+pp)r}{qq}+\frac{(1+pp)^2r^2}{q^4}\Big)=\int dx\Big(\frac{9pp(1+pp)^2}{q}-\frac{6(1+pp)^3r}{q^3}+\frac{(1+pp)^4r^2}{q^5}\Big),$$

ex quo Z erit functio ipſarum p, q & r; unde differentiando prodibit:

$$dZ=\frac{18pdp(1+pp)(1+3pp)}{q}-\frac{9ppdq(1+pp)^2}{qq}-\frac{6dr(1+pp)^3}{q^3}$$
$$-\frac{36prdp(1+pp)^2}{q^3}+\frac{18rdq(1+pp)^3}{q^4}+\frac{2rdr(1+pp)^4}{q^5}$$
$$+\frac{8rrpdp(1+pp)^3}{q^5}-\frac{5r^2dq(1+pp)^4}{q^6}$$

Comparatione ergo cum forma generali inſtituta, erit $M=0$; $N=0$;

$$Z=\frac{9pp(1+pp)^2}{q}-\frac{6(1+pp)^3r}{q^3}+\frac{(1+pp)^4r^2}{q^5}$$
$$P=\frac{18p(1+pp)(1+3pp)}{q}-\frac{36pr(1+pp)^2}{q^3}+\frac{8rrp(1+pp)^3}{q^5}$$
$$Q=-\frac{9pp(1+pp)^2}{qq}+\frac{18r(1+pp)^3}{q^4}-\frac{5rr(1+pp)^4}{q^6}$$
$$R=-\frac{6(1+pp)^3}{q^3}+\frac{2r(1+pp)^4}{q^5}$$

Cum nunc ſit $M=0$ & $N=0$, ſolutio cadit in Caſum quintum, eritque æquatio pro curva quæſita hæc,

$$0=Ap-B-Z+Qq+Rr-\frac{qdR}{dx}$$

quæ, factis ſubſtitutionibus, tranſit in hanc

$$0=Ap-B-\frac{18pp(1+pp)^2}{q}-\frac{16pr(1+pp)^3}{q^3}+\frac{6rr(1+pp)^4}{q^5}$$
$$-\frac{2dr(1+pp)^4}{q^4dx}+\frac{36p(1+pp)^2}{q};$$

quæ æquatio nimis eſt complicata, quam ut ejus ulteriores integrationes ſuſcipi queant. Cæterum apparet hanc æquationem eſſe differentialem quarti ordinis, ita ut per quatuor reſiduas integrationes quatuor conſtantes adhuc ingrediantur: ex quo ſex data oportebit eſſe puncta, per quæ curva tranſeat, ut Problema determinetur.

CAPUT

Digitized by Google

CAPUT III.

De inventione curvarum maximi minimive proprietate præditarum, ſi in ipſa maximi minimive formula inſunt quantitates indeterminatæ.

PROPOSITIO I. PROBLEMA.

1. *Invenire incrementa, quæ quantitas integralis indeterminata, in quovis abſciſſæ puncto, ab aucta alicubi una applicata* Nn *particula* nν, *capit.* Fig. 4.

SOLUTIO.

Sit abſciſſa AH $=x$, applicata reſpondens Hh $=y$, & propoſita ſit quantitas quæcunque indeterminata Π, abſciſſæ AH reſpondens, quæ ſit formula integralis indefinite integrationem non admittens. Quantitas hæc Π ita ſit comparata, ut ipſa, quatenus abſciſſæ AH ſeu puncto H reſpondet, ab aucta applicata Nn non mutetur: quod eveniet, ſi in Π differentialia non ultra quintum gradum aſſurgant; quem in finem quintam demum applicatam Nn ab Hh computando mutari ponimus. Si enim in Π differentialia altiorum graduum continerentur, tum deberet ulterior demum applicata poſt Nn particula infinite parva augeri. Sufficiet autem ſolutionem ad quinque tantum differentialium in Π contentorum gradus extendere; cum inde, ſi etiam altiora affuerint differentialia, ſolutionem ad ea accommodare liceat. Quemadmodum igitur puncto abſciſſæ H reſpondet valor Π, ita ſecundum noſtram notandi methodum, puncto ſequenti I reſpondebit valor Π', puncto K vero Π'', puncto L valor Π''', & ita porro. Id ergo erit inveſtigandum, quanta incrementa ex tranſlatione puncti n in ν ſinguli hi valores derivativi Π', Π'', Π''', Π^{IV}, &c. accipiant, ſeu definiri debent

bent eorum differentialia, ſi ſola applicata Nn, quæ eſt $= y^{V}$ variari & particula $n\nu$ augeri ponatur: erit autem hoc ſenſu $d.\,\Pi = 0$, quia valorem Π puncto H reſpondentem inde non affici ponimus. Quoniam jam Π eſt formula integralis indefinita, ſit ea $= \int [Z]\,dx$, & $[Z]$ ſit functio ipſarum x, y, p, q, r, s & t, ita ut ſit $d[Z] = [M]\,dx + [N]\,dy + [P]\,dp + [Q]\,dq + [R]\,dr + [S]\,ds + [T]\,dt$; unde ſimul valores derivativi ipſius $d[Z]$, nempe $d[Z']$, $d[Z'']$, $d[Z''']$, &c. per notandi modum receptum formari poterunt. His poſitis, erit ut ſequitur

$$\Pi = \int [Z]\,dx$$
$$\Pi' = \int [Z]\,dx + [Z]\,dx$$
$$\Pi'' = \int [Z]\,dx + [Z]\,dx + [Z']\,dx$$
$$\Pi''' = \int [Z]\,dx + [Z]\,dx + [Z']\,dx + [Z'']\,dx$$
$$\Pi^{IV} = \int [Z]\,dx + [Z]\,dx + [Z']\,dx + [Z'']\,dx + [Z''']\,dx$$
&c.

Jam videamus quanta incrementa ſingula hæc membra $[Z]\,dx$, $[Z']\,dx$, $[Z'']\,dx$, $[Z''']\,dx$, &c. ex adjecta particula $n\nu$ ad applicatam Nn capiant; quæ obtinebuntur ex ipſorum differentialibus, ponendo loco differentialium valores §. 56 Capitis præcedentis expoſitos: erit itaque

$$d.\,[Z]\,dx = n\nu.\,dx.\,\frac{[T]}{dx^5}$$
$$d.\,[Z']\,dx = n\nu.\,dx\left(\frac{[S']}{dx^4} - \frac{5[T']}{dx^5}\right)$$
$$d.\,[Z'']\,dx = n\nu.\,dx\left(\frac{[R'']}{dx^3} - \frac{4[S'']}{dx^4} + \frac{10[T'']}{dx^5}\right)$$
$$d.\,[Z''']\,dx = n\nu.\,dx\left(\frac{[Q''']}{dx^2} - \frac{3[R''']}{dx^3} + \frac{6[S''']}{dx^4} - \frac{10[T''']}{dx^5}\right)$$
$$d.\,[Z^{IV}]\,dx = n\nu.\,dx\left(\frac{[P^{IV}]}{dx} - \frac{2[Q^{IV}]}{dx^2} + \frac{3[R^{IV}]}{dx^3} - \frac{4[S^{IV}]}{dx^4} + \frac{5[Z^{IV}]}{dx^5}\right)$$
$$d.\,[Z^{V}]\,dx = n\nu.\,dx\left([N^{V}] - \frac{[P^{V}]}{dx} + \frac{[Q^{V}]}{dx^2} - \frac{[R^{V}]}{dx^3} + \frac{[S^{V}]}{dx^4} - \frac{[T^{V}]}{dx^5}\right)$$
$$d.\,[Z^{VI}]\,dx = 0,$$
$d.\,[Z^{VII}]\,dx = 0$. & reliqua ſequentia omnia evaneſcent.

Ex

Ex his nunc colligentur incrementa valorum Π, Π', Π'', Π''', &c. quæ recipiunt ex tranſlatione puncti n in ν; erit ſcilicet

$$d.\,\Pi = 0$$

$$d.\,\Pi' = n\nu.\,dx.\,\frac{[T]}{dx^5}$$

$$d.\,\Pi'' = n\nu.\,dx\left(\frac{[S']}{dx^4} - \frac{4[T'] + d[T]}{dx^5}\right)$$

$$d.\,\Pi''' = n\nu.\,dx\left(\frac{[R'']}{dx^3} - \frac{3[S''] + d[S']}{dx^4} + \frac{6[T''] + 4d[T'] - d[T]}{dx^5}\right)$$

$$d.\,\Pi^{IV} = n\nu.\,dx\left(\frac{[Q''']}{dx^2} - \frac{2[R'''] + d[R'']}{dx^3} + \frac{3[S'''] + 3d[S''] - d[S']}{dx^4} - \frac{4[T'''] + 6d[T''] - 4d[T'] + [dT]}{dx^5}\right)$$

$$d.\,\Pi^{V} = n\nu.\,dx\left(\frac{[P^{IV}]}{dx} - \frac{[Q^{IV}] + d[Q''']}{dx^2} + \frac{[R^{IV}] + 2d[R'''] - d[R'']}{dx^3} - \frac{[S^{IV}] + 3d[S'''] - 3d[S''] + d[S']}{dx^4} + \frac{[T^{V}] + 4d[T'''] - 6d[T''] + 4d[T'] - d[T]}{dx^5}\right)$$

$$d.\,\Pi^{VI} = n\nu.\,dx\left([N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{d[Q^{V}] - d[Q''']}{dx^2} - \frac{d[R^{V}] - 2d[R'''] + d[R'']}{dx^3} + \frac{d[S^{V}] - 3d[S'''] + 3d[S''] - d[S']}{dx^4} - \frac{d[T^{V}] - 4d[T'''] + 6d[T''] - 4d[T'] + d[T]}{dx^5}\right)$$

Huic autem incremento æqualia ſunt incrementa omnium ſequentium valorum, nempe ipſorum Π^{VII}, Π^{VIII}, Π^{IX}, &c. Atqui valoris Π^{VI} & omnium ſequentium incrementum idem erit $= n\nu.\,dx\left([N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{dd[Q''']}{dx^2} - \frac{d^3[R'']}{dx^3} + \frac{d^4[S']}{dx^4} - \frac{d^5[T]}{dx^5}\right)$. Poterunt autem hæc incrementa ad eadem ſigna reduci, reſpectu litterarum $[P]$, $[Q]$, $[R]$, $[S]$, & $[T]$, ſicque prodibit

bent eorum differentialia, ſi ſola applicata Nn, quæ eſt $= y^{v}$ variari & particula $n\nu$ augeri ponatur: erit autem hoc ſenſu $d.\,\Pi = 0$, quia valorem Π puncto H reſpondentem inde non affici ponimus. Quoniam jam Π eſt formula integralis indefinita, ſit ea $= \int [Z]\,dx$, & $[Z]$ ſit functio ipſarum x, y, p, q, r, s & t, ita ut ſit $d[Z] = [M]\,dx + [N]\,dy + [P]\,dp + [Q]\,dq + [R]\,dr + [S]\,ds + [T]\,dt$; unde ſimul valores derivativi ipſius $d[Z]$, nempe $d[Z']$, $d[Z'']$, $d[Z''']$, &c. per notandi modum receptum formari poterunt. His poſitis, erit ut ſequitur

$$\Pi = \int [Z]\,dx$$
$$\Pi' = \int [Z]\,dx + [Z]\,dx$$
$$\Pi'' = \int [Z]\,dx + [Z]\,dx + [Z']\,dx$$
$$\Pi''' = \int [Z]\,dx + [Z]\,dx + [Z']\,dx + [Z'']\,dx$$
$$\Pi^{IV} = \int [Z]\,dx + [Z]\,dx + [Z']\,dx + [Z'']\,dx + [Z''']\,dx$$

&c.

Jam videamus quanta incrementa ſingula hæc membra $[Z]\,dx$, $[Z']\,dx$, $[Z'']\,dx$, $[Z''']\,dx$, &c. ex adjecta particula $n\nu$ ad applicatam Nn capiant; quæ obtinebuntur ex ipſorum differentialibus, ponendo loco differentialium valores §. 56 Capitis præcedentis expoſitos: erit itaque

$$d.\,[Z]\,dx = n\nu.\,dx.\,\frac{[T]}{dx^5}$$
$$d.\,[Z']\,dx = n\nu.\,dx\left(\frac{[S']}{dx^4} - \frac{5[T']}{dx^5}\right)$$
$$d.\,[Z'']\,dx = n\nu.\,dx\left(\frac{[R'']}{dx^3} - \frac{4[S'']}{dx^4} + \frac{10[T'']}{dx^5}\right)$$
$$d.\,[Z''']\,dx = n\nu.\,dx\left(\frac{[Q''']}{dx^2} - \frac{3[R''']}{dx^3} + \frac{6[S''']}{dx^4} - \frac{10[T''']}{dx^5}\right)$$
$$d.\,[Z^{IV}]\,dx = n\nu.\,dx\left(\frac{[P^{IV}]}{dx} - \frac{2[Q^{IV}]}{dx^2} + \frac{3[R^{IV}]}{dx^3} - \frac{4[S^{IV}]}{dx^4} + \frac{5[Z^{IV}]}{dx^5}\right)$$
$$d.\,[Z^{V}]\,dx = n\nu.\,dx\left([N^{V}] - \frac{[P^{V}]}{dx} + \frac{[Q^{V}]}{dx^2} - \frac{[R^{V}]}{dx^3} + \frac{[S^{V}]}{dx^4} - \frac{[T^{V}]}{dx^5}\right)$$
$$d.\,[Z^{VI}]\,dx = 0,$$

$d.\,[Z^{VII}]\,dx = 0$. & reliqua ſequentia omnia evaneſcent.

Ex

Ex his nunc colligentur incrementa valorum Π, Π', Π'', Π''', &c. quæ recipiunt ex tranſlatione puncti n in ν; erit ſcilicet

$$d.\,\Pi = 0$$

$$d.\,\Pi' = n\nu.\,dx.\,\frac{[T]}{dx^5}$$

$$d.\,\Pi'' = n\nu.\,dx\left(\frac{[S']}{dx^4} - \frac{4[T'] + d[T]}{dx^5}\right)$$

$$d.\Pi''' = n\nu.\,dx\left(\frac{[R'']}{dx^3} - \frac{3[S''] + d[S']}{dx^4} + \frac{6[T''] + 4d[T'] - d[T]}{dx^5}\right)$$

$$d.\Pi^{IV} = n\nu.\,dx\left(\frac{[Q''']}{dx^2} - \frac{2[R'''] + d[R'']}{dx^3}\right.$$
$$\left. + \frac{3[S'''] + 3d[S''] - d[S']}{dx^4} - \frac{4[T'''] + 6d[T''] - 4d[T'] + [dT]}{dx^5}\right)$$

$$d.\Pi^{V} = n\nu.dx\left(\frac{[P^{IV}]}{dx} - \frac{[Q^{IV}] + d[Q''']}{dx^2} + \frac{[R^{IV}] + 2d[R'''] - d[R'']}{dx^3} - \right.$$
$$\left. \frac{[S^{IV}] + 3d[S'''] - 3d[S''] + d[S']}{dx^4} + \frac{[T^{IV}] + 4d[T'''] - 6d[T''] + 4d[T'] - d[T]}{dx^5}\right)$$

$$d.\Pi^{VI} = n\nu.\,dx\left([N^{V}] - \frac{d[P^{V}]}{dx} + \frac{d[Q^{V}] - d[Q''']}{dx^2}\right.$$
$$- \frac{d[R^{IV}] - 2d[R'''] + d[R'']}{dx^3} + \frac{d[S^{IV}] - 3d[S'''] + 3d[S''] - d[S']}{dx^4}$$
$$\left. - \frac{d[T^{IV}] - 4d[T'''] + 6d[T''] - 4d[T'] + d[T]}{dx^5}\right)$$

Huic autem incremento æqualia ſunt incrementa omnium ſequentium valorum, nempe ipſorum Π^{VII}, Π^{VIII}, Π^{IX}, &c. Atqui valoris Π^{VI} & omnium ſequentium incrementum idem erit $= n\nu.\,dx\left([N'] - \frac{d[P^{V}]}{dx} + \frac{dd[Q''']}{dx^2} - \frac{d^3[R'']}{dx^3} + \frac{d^4[S']}{dx^4} - \frac{d^5[T]}{dx^5}\right)$. Poterunt autem hæc incrementa ad eadem ſigna reduci, reſpectu litterarum $[P]$, $[Q]$, $[R]$, $[S]$, & $[T]$, ſicque prodibit

Digitized by Google

$$d.\Pi = 0$$

$$d.\Pi' = n\nu.\, dx.\, \frac{[T]}{dx^5}$$

$$d.\Pi'' = n\nu.\, dx \left(\frac{[S']}{dx^4} - \frac{4[T] + 5d[T]}{dx^5} \right)$$

$$d.\Pi''' = n\nu.\, dx \left(\frac{[R'']}{dx^3} - \frac{3[S'] + 4d[S']}{dx^4} + \frac{6[T] + 15d[T] + 10dd[T]}{dx^5} \right)$$

$$d.\Pi^{IV} = n\nu.\, dx \left(\frac{[Q''']}{dx^2} - \frac{2[R''] + 3d[R'']}{dx^3} + \frac{3[S'] + 8d[S'] + 6dd[S']}{dx^4} \right.$$
$$\left. - \frac{4[T] + 15d[T] + 20dd[T] + 10d^3[T]}{dx^5} \right)$$

$$d.\Pi^{V} = n\nu.\, dx \left(\frac{[P^{IV}]}{dx} - \frac{[Q'''] + 2d[Q''']}{dx^2} + \frac{[R''] + 3d[R''] + 3dd[R'']}{dx^3} \right.$$
$$\left. - \frac{[S']4d + [S'] + 6dd[S'] + 4d^3[S']}{dx^4} + \frac{[T] + 5d[T] + 10dd[T] + 10d^3[T] + 5d^4[T]}{dx^5} \right)$$

$$d\Pi^{VI} = n\nu.\, dx \left([N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{dd[Q''']}{dx^2} - \frac{d^3[R'']}{dx^3} + \frac{d^4[S']}{dx^4} - \frac{d^5[T]}{dx^5} \right)$$

cui ſequentium valorum omnium incrementa ſunt æqualia. *Q. E. I.*

C o r o l l. I.

2. Si ergo Π fuerit hujuſmodi quantitas indeterminata, ſeu formula integralis indefinite integrationem non admittens, tum ejus omnes valores poſt locum abſciſſæ, ubi una applicata augeri concipitur, mutationem patientur, & aliquot ejus etiam valores ante illum locum, quorum numerus pendet a gradu differentialium, quæ in ea formula Π inſunt.

C o r o l l. II.

3. Quod ſi ergo iſtiuſmodi quantitas inſit in maximi minimive formula $\int Z\, dx$, tum ejus valor differentialis non ſolum ab aliquot abſciſſæ elementis, verum a tota abſciſſa, cui maximum minimumve reſpondere debet, pendebit.

C o-

C o r o l l. III.

4. His igitur caſibus abſciſſam illam, pro qua maximum minimumve quæritur, determinatam eſſe oportet, atque curva quæ, pro hac abſciſſa, maximi minimive proprietate gaudere reperta fuerit, eadem pro aliis abſciſſis hac proprietate non erit prædita.

S c h o l i o n.

5. Mox clarius diſcrimen, quod intercedit inter quæſtiones; in quibus Z eſt quantitas vel determinata vel indeterminata, perſpicietur; quando Problemata hujus generis ſumus tractaturi. Pluribus modis autem tales quæſtiones poſſunt variari, prout in maximi minimive formula $\int Z\,dx$, quantitas Z vel tantum eſt functio ejuſmodi formulæ indeterminatæ Π, qualem contemplati ſumus, vel inſuper quantitates determinatas, x, y, p, q, r, s, &c. comprehendit. Deinde in Z etiam ineſſe poterunt plures ejuſmodi formulæ integrales indefinitæ a ſe invicem diverſæ. Ad hos autem diverſos caſus una regula, ſuperioribus jam traditis addita, ſufficere poterit. Præcipuum autem momentum poſitum eſt in ipſa formula indeterminata $\Pi = \int [Z]\,dx$, pro qua hîc poſuimus eſſe $[Z]$ functionem determinatam; quod ſi autem hæc ipſa quantitas $[Z]$ denuo ejuſmodi formulas integrales indefinitas complectatur, iterum peculiari ſolutione erit opus. Quin etiam iſta complicatio formularum indeterminatarum in infinitum poteſt extendi; id quod eveniet ſi quantitas $[Z]$ denuo in ſe complectatur ipſam quantitatem Π, ita ut ſit $d[Z] = [L]\,d\Pi + [M]\,dx + [N]\,dy + [P]\,dp + [Q]\,dq + [R]\,dr +$ &c. tum enim ob $d\,\Pi = [Z]\,dx$, iterum conſiderari oportebit valorem $d[Z] = [L]\,d\,\Pi + [M]\,dx +$ &c. hicque progreſſus in infinitum continuabitur. Hinc autem methodus naſcetur ea reſolvendi Problemata, in quibus curva quæritur maximum minimumve habens valorem formulæ $\int Z\,dx$, quando quantitas Z non datur,

ut:

Digitized by Google

ut hactenus, ſive determinate ſive indeterminate, ſed tantum per æquationem differentialem, cujus integratio omnino non poteſt abſolvi: cujuſmodi quæſtio eſt, ſi quæratur curva, in qua minimum ſit expreſſio $\int \frac{dx\sqrt{(1+pp)}}{\sqrt{v}}$, exiſtente $dv = gdx - hv^n dx\sqrt{(1+pp)}$: atque ejuſmodi quæſtionum reſolutionem in hoc Capite quoque trademus.

PROPOSITIO II. PROBLEMA.

Fig. 4. 6. *Si* Z *fuerit functio quantitatis indeterminatæ* n, *ita ut ſit* $dZ = Ldn$, *ſitque* $n = \int [Z]\,dx$, *exiſtente* $d[Z] = [M]dx + [N]dy + [P]dp + [Q]dq + [R]dr +$ &c. *invenire curvam* az *quæ pro data abſciſſa* AZ *habeat valorem formulæ* $\int Z\,dx$ *maximum vel minimum.*

SOLUTIO.

Poſita abſciſſa $AH = x$, applicata $Hh = y$, ſit tota abſciſſa AZ, cui maximum minimumve reſpondere debet, $= a$, diviſo igitur ſpatio HZ in elementa innumera infinite parva HI, IK, KL, LM, &c. debebit eſſe $\int Z\,dx + Z\,dx + Z'dx + Z''\,dx + Z'''\,dx +$ &c. donec ad extremum punctum Z perveniatur, maximum minimumve. Ad hoc efficiendum, quærendi ſunt valores differentiales quos ſinguli hi termini a translatione puncti n in ν accipiunt, quorum ſumma, nihilo æqualis poſita, dabit æquationem pro curva quæſita. Quoniam autem mutationem ab $n\nu$ oriundam non ultra H verſus A porrigi ponimus, erit termini $\int Z\,dx$ valor differentialis nullus. Reliquorum terminorum valores differentiales reperientur, ſi ii differentientur, atque in differentialibus ſcribantur ea incrementa, quæ in Propoſitione præcedente invenimus, ex translatione puncti n in ν oriri. Erit autem

d. Z dx

Digitized by Google

$$d.\ Z\,dx = L\,dx.\,dn$$
$$d.\ Z'\,dx = L'\,dx.\,dn'$$
$$d.\ Z''\,dx = L''\,dx.\,dn''$$
$$d.\ Z'''\,dx = L'''\,dx.\,dn'''$$
$$d.\ Z^{IV}\,dx = L^{IV}\,dx.\,dn^{IV}$$

Quodſi jam loco differentialium dn, dn', dn'', dn''', &c. valores ſupra inventos ex translatione puncti n in ν ortos ſubſtituamus obtinebimus.

$$d.\,Z\,dx = 0.$$

$$d.\,Z'\,dx = n\nu.\,L'dx^2.\,\frac{[T]}{dx^5}$$

$$d.\,Z''dx = n\nu.L''dx^2\left(\frac{[S']}{dx^4} - \frac{4[T] + 5d[T]}{dx^5}\right)$$

$$d.Z'''dx = n\nu.L'''dx^2\left(\frac{[R'']}{dx^3} - \frac{3[S'] + 4d[S']}{dx^4} + \frac{6[T] + 15d[T] + 10dd[T]}{dx^5}\right)$$

$$d.\,Z^{IV}dx = n\nu.L^{IV}dx^2\left(\frac{[Q''']}{dx^2} - \frac{2[R''] + 3d[R'']}{dx^3} + \frac{3[S'] + 8d[S'] + 6dd[S']}{dx^4} - \frac{4[T] + 15d[T] + 20dd[T] + 10d^3[T]}{dx^5}\right)$$

$$d.Z^{V}dx = n\nu.\,L^{V}dx^2\left(\frac{[P^{IV}]}{dx} - \frac{[Q'''] + 2d[Q''']}{dx^2} + \frac{[R''] + 3d[R''] + 3dd[R'']}{dx^3} - \frac{[S'] + 4d[S'] + 6dd[S'] + 4d[S']}{dx^4} + \frac{[T] + 5d[T] + 10dd[T] + 10d^3[T] + 5d^4[T]}{dx^5}\right)$$

$$d.Z^{VI}dx = n\nu.L^{VI}dx^2\left([N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{dd[Q''']}{dx^2} - \frac{d^3[R'']}{dx^3} + \frac{d^4[S']}{dx^4} - \frac{d^5[T]}{dx^5}\right)$$

$$d.Z^{VII}dx = n\nu.L^{VII}dx^2\left([N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{dd[Q''']}{dx^2} - \frac{d^3[R'']}{dx^3} + \frac{d^4[S']}{dx^4} - \frac{d^5[T]}{dx^5}\right)$$

&c.

Sequentium ſcilicet terminorum incrementa eadem hac lege progrediuntur. Addantur jam ſenorum priorum terminorum incrementa, prodibit terminorum $Z\,dx + Z'\,dx + Z''\,dx + Z'''dx + Z^{IV}\,dx + Z^{V}\,dx$ incrementum totale $=$

$$n\nu.dx^2\left(\frac{L^{V}[P^{IV}]}{dx} - \frac{[Q''']dL^{IV} + 2L^{IV}d[Q''']}{dx^2} + \frac{[R'']ddL''' + 3d[R'']dL''' + 3L'''dd[R'']}{dx^3} - \frac{[S']d^3L'' + 4d[S']ddL'' + 6dL''dd[S'] + 4L''d^3[S']}{dx^4} \ldots\ldots\right.$$

$$+ \frac{[T]d^4L' + 5d[T]d^3L' + 10dd[T]ddL' + 10dL'd^3[T] + 5L'd^4[T]}{dx^5})$$

in qua expreſſione, quia omnes termini inter ſe ſunt homogenei, jam indices numerici negligi poterunt. Sequentium autem terminorum $L^{VI}dx + L^{VII}dx +$ &c. omnium incrementum erit $=$

$$nv.\,dx([N^{V}] - \frac{d[P^{IV}]}{dx} + \frac{dd[Q''']}{dx^2} - \frac{d^3[R'']}{dx^3} + \frac{d^4[S']}{dx^4} - \frac{d^5[T]}{dx^5})$$

$(L^{VI}dx + L^{VII}dx + L^{VIII}dx + L^{IX}dx +$ &c. uſque in Z). Hic autem poſterior factor definietur per integrationem formulæ $\int L\,dx$, quæ reſpondet abſciſſæ indefinitæ AH $= x$; ponatur in hac formula poſt integrationem $x = a$, abeatque ea in H, erit H valor formulæ $\int L\,dx$ abſciſſæ toti propoſitæ AZ reſpondens; a qua ergo ſi auferatur $\int L\,dx$, remanebit $H - \int L\,dx$ valor portioni HZ vel NZ reſpondens, qui ergo loco $L^{VI}dx + L^{VII}dx + L^{VIII}dx +$ &c. ſubſtitui poteſt. Quamobrem tandem formulæ $\int Z\,dx$ valor differentialis toti abſciſſæ AZ reſpondens erit $=$

$$nv.\,dx(H - \int L\,dx)([N] - \frac{d[P]}{dx} + \frac{dd[Q]}{dx^2} - \frac{d^3[R]}{dx^3} + \frac{d^4[S]}{dx^4} - \frac{d^5[T]}{dx^5})$$

$$+ nv.\,dx(L[P] - \frac{[Q]dL + 2Ld[Q]}{dx} + \frac{[R]ddL + 3d[R]dL + 3Ldd[R]}{dx^2}$$

$$- \frac{[S]d^3L + 4d[S]ddL + 6dLdd[S] + 4Ld^3[S]}{dx^3} \ldots$$

$$+ \frac{[T]d^4L + 5d[T]d^3L + 10dd[T]ddL + 10dLd^3[T] + 5Ld^4[T]}{dx^4})$$

qui ad hanc formam commodiorem reduci poteſt, ut ſit $=$

$$nv.\,dx([N](H - \int L\,dx) - \frac{d.[P](H - \int L\,dx)}{dx} + \frac{dd.[Q](H - \int L\,dx)}{dx^2}$$

$$- \frac{d^3.[R](H - \int L\,dx)}{dx^3} + \frac{d^4.[S](H - \int L\,dx)}{dx^4} - \frac{d^5[T](H - \int L\,dx)}{dx^5});$$

qui valor differentialis, quouſque occaſio poſtulabit, ulterius continuari poterit: is autem, nihilo æqualis poſitus, dabit æquationem pro curva quæſita. Q. E. I.

COROLL. I.

7. Quoniam $H - \int L\,dx$ eſt valor formulæ $\int L\,dx$ reſpondens abſciſſæ portioni $AZ = a - x$, ſi ponatur $AZ = a - x = u$, erit $\int L\,du$ ille ipſe valor $H - \int L\,dx$, quo opus eſt; ſiquidem $\int L\,du$ ita integretur, ut evaneſcat poſito $u = 0$.

COROLL. II.

8. Quodſi igitur abſciſſarum initium capiatur in puncto Z; ita ut abſciſſa ZH ponatur $= u$, utque ubique ponatur $x = a - u$, prodibit æquatio pro curva inter coordinatas u & y; hujuſque curvæ ea portio quæſito ſatisfaciet, quæ reſpondet abſciſſæ $AZ = a$. Interim notandum eſt cum in ipſa maximi minimive formula $\int Z\,dx$, tum in $\int [Z]\,dx$, abſciſſarum initium in puncto A capi debere.

COROLL. III.

9. Si ergo quæratur curva ad datam abſciſſam AZ relata; in qua maximum minimumve debeat eſſe $\int Z\,dx$; ſitque Z functio quæcunque ipſius $\Pi = \int [Z]\,dx$; exiſtente $dZ = L\,d\Pi$ & $d[Z] = [M]\,dx + [N]\,dy + [P]\,dp + [Q]\,dq + [R]\,dr + \&c.$ habebitur pro curva quæſita iſta æquatio:

$$0 = [N]\int L\,du - \frac{d.[P]\int L\,du}{dx} + \frac{dd.[Q]\int L\,du}{dx^2} - \frac{d^3[R]\int L\,du}{dx^3} + \&c.$$

ubi eſt $u = a - x$, & $\int L\,du$ denotat valorem formulæ $\int L\,dx$ portioni abſciſſæ $HZ = u$ reſpondentem.

COROLL. IV.

10. Poſſunt ergo vel bina abſciſſarum initia A & Z, binæque abſciſſæ $AH = x$, & $ZH = u$ conſiderari, quarum illa in integrali $\int [Z]\,dx$ ſeu Π, hæc vero in integrali $\int L\,dx$ ſpectari debet, vel unica tantum abſciſſa $AH = x$; quo caſu, loco

Digitized by Google

$\int L\,du$ ſcribi debet $H - \int L\,dx$; denotante H valorem, quem præbet formula $\int L\,dx$, poſito $x = AH = a$.

COROLL. V.

11. Quia Z eſt functio ipſius Π tantum, ita ut nullas alias quantitates variabiles in ſe complectatur, ob $dZ = L\,d\Pi$, erit etiam L functio ipſius Π tantum.

COROLL. VI.

12. Si $[Z]$ eſſet functio ipſius x tantum; tum foret $\Pi = \int [Z]\,dx$ quantitas determinata, atque functio ipſius x, hincque etiam Z; ex quo maximum minimumve non inveniet locum. Idem oſtendit ſolutio; fiet enim $[N] = 0$, $[R] = 0$ &c. atque æquatio abit in identicam $0 = 0$.

SCHOLION I.

13. Occurrunt hic nonnulli primarii caſus conſiderandi, quorum primus eſt, ſi fuerit $[Z]$ functio ipſarum x & y tantum, ita ut ſit $d[Z] = [M]\,dx + [N]\,dy$. Quod ſi nunc quæratur curva in qua maximum minimumve ſit formula $\int Z\,dx$ pro data abſciſſa $AZ = a$, exiſtente Z functione quacunque ipſius $\int [Z]\,dx = \Pi$, ita ut ſit $dZ = L\,d\Pi$; habebitur pro curva quæſita iſta æquatio $0 = [N]\,(H - \int L\,dx)$; erit ergo vel $[N] = 0$ vel $H = \int L\,dx$, ſeu $L = 0$; quarum æquationum ſi vel altera vel utraque præbeat lineam curvam, ea non ſolum ſatisfaciet Problemati pro abſciſſa $AZ = a$, ſed etiam pro alia quacunque abſciſſa indefinita x: id quod inde colligitur, quod ex æquatione, quantitas H, quæ pendet ab abſciſſa determinata a, ex calculo exceſſerit. Quod autem ſpeciatim ad æquationem $L = 0$ attinet: quia L eſt functio ipſius Π ſeu $\int [Z]\,dx$, fiet $\int [Z]\,dx =$ conſt. determinatæ, quod niſi ſit $[Z] = 0$, fieri nequit: binæ igitur æquationes hoc caſu ſatisfacientes, erunt $[N] = 0$, atque $[Z] = 0$.

SCHO-

SCHOLION II.

14. Deinde vero conſiderari meretur caſus quo $[N]$ evaneſcit; id quod evenit, ſi $[Z]$ fuerit functio ipſarum x, p, q, r, &c. non involvens y. Ponamus eſſe $[Z]$ functionem ipſarum x & p, atque $d[Z] = [M]dx + [P]dp$. Si igitur ponatur $\int [Z]dx = \Pi$, atque curva quæratur, in qua, pro abſciſſa definita $AZ = a$, maximum minimumve ſit formula $\int Z dx$, exiſtente Z functione ipſius Π, ita ut ſit $dZ = L d\Pi$; orietur pro curva quæſita iſta æquatio $0 = -\frac{d.[P](H - \int L dx)}{dx}$; ideoque $Conſt. = [P](H - \int L dx)$. Hæc vero conſtans, per integrationem ingreſſa, non eſt arbitraria; nam eam ita comparatam eſſe oportet, ut poſito $x = a$, quo caſu ſit $\int L dx = H$, fiat $\frac{Conſt.}{[P]} = 0$. Hoc autem evenire non poteſt, niſi vel hæc conſtans ponatur $= 0$, vel quantitas $[P]$ ita comparata ſit ut fiat $= \infty$, poſito $x = a$. Priori caſu habetur vel $[P] = 0$, vel $\int L dx = H$, hoc eſt $L = 0$, ſeu $\int [Z]dx = Conſt.$ ſeu potius $[Z] = 0$; poſteriori caſu autem, conſtans tamen pro arbitrio non accipi poteſt, nam determinabitur, ponendo $x = a - dx$, eo modo, quo expreſſiones quæ certis caſibus indeterminatæ videntur definiri ſolent. Atque hinc perſpicitur in hujuſmodi Problematis numerum conſtantium arbitrariarum in ſolutionem ingredientium, cui æqualis ſumi debet numerus punctorum, per quæ curvæ ſatisfacienti tranſeundum eſt, non ex gradu differentialium judicari poſſe. Pervenietur enim ſæpe, tollendo per differentiationem omnes formulas integrales, ad æquationem differentialem altioris gradus, a quo nequaquam Problematis determinatio per aliquot puncta pendebit.

EXEMPLUM I.

15. *Si denotet* Π *aream curvæ* $\int y\,dx$, *atque* Z *ſit functio quæcunque ipſius* Π, *invenire curvam quæ, pro data abſciſſa* $= a$, *habeat valorem formulæ* $\int Z\,dx$ *maximum vel minimum.*

 Quia

Quia eſt Z functio ipſius Π; ſit $dZ = L\,d\Pi$, erit L functio ipſius $\Pi = \int y\,dx$. Deinde cum ſit $d\Pi = y\,dx$; erit $[Z] = y$, & ob $d[Z] = [M]dx + [N]dy + [P]dp +$ &c. fiet $[M] = 0$, $[N] = 1$, $[P] = 0$, $[Q] = 0$, &c. unde pro curva quæſita hæc habebitur æquatio $0 = H - \int L\,dx$; ideoque $L = 0$. Hinc erit $\Pi = \int y\,dx =$ conſtanti cuidam, porroque $y = 0$. Satisfacit ergo ſola linea recta in ipſum axem incidens; idque pro quacunque abſciſſa æque ac pro definita $= a$.

EXEMPLUM II.

16. *Si* Π *denotet arcum curvæ* $= \int dx\sqrt{(1+pp)}$ *ejuſque functio quæcunque fuerit* Z; *invenire curvam, quæ, pro data abſciſſa* AZ $= a$, *habeat valorem formulæ* $\int Z\,dx$ *maximum vel minimum.*

Ob $dZ = L\,d\Pi$, erit L functio ipſius arcus Π; & ob $d\Pi = dx\sqrt{(1+pp)}$, erit $[Z] = \sqrt{(1+pp)}$ & $[M] = 0$, $[N] = 0$, $[P] = \frac{p}{\sqrt{(1+pp)}}$, $[Q] = 0$, &c. unde pro curva quæſita iſta habebitur æquatio: $0 = -d.\frac{p}{dx\sqrt{(1+pp)}}(H - \int L\,dx)$; hincque $C = \frac{p}{\sqrt{(1+pp)}}(H - \int L\,dx)$: ubi conſtans C ita determinari debet, ut, poſito $x = a$, fiat $C = \frac{p}{\sqrt{(1+pp)}} \times 0$; quare quia $\frac{p}{\sqrt{(1+pp)}}$ infinitum fieri nequit, neceſſe eſt ut ſit $C = 0$; ideoque vel $\frac{p}{\sqrt{(1+pp)}} = 0$; vel $\int L\,dx = H$. Fiet ergo, ex poſteriore æquatione, $L = 0$, & $\Pi =$ conſtanti cuidam: ex quo porro deducitur $d\Pi = dx\sqrt{(1+pp)} = 0$, cui conditioni nullo modo ſatisfieri poteſt. Ex priore æquatione autem deducitur $p = 0$, ſeu $dy = 0$, quæ eſt æquatio pro linea recta axi AZ parallela, quæ quæſtioni pro abſciſſa quacunque ſatisfacit.

Digitized by Google

EXEMPLUM III.

17. *Denotet* Π *ſuperficiem ſolidi rotundi ex converſione curvæ* a h *circa axem* A Z *orti, quæ eſt ut* $\int y\,dx\sqrt{(1+pp)}$, *hujuſque ſuperficiei functio ſit quacunque* Z, *invenire curvam, in qua pro data abſciſſa* A Z = a, *maximum minimumve ſit* $\int Z\,dx$.

Ob $dZ = L\,d\Pi$, erit L functio ipſius $\Pi = \int y\,dx\sqrt{(1+pp)}$; & ob $d\Pi = y\,dx\sqrt{(1+pp)}$ fiet $[Z] = y\sqrt{(1+pp)}$, & $d[Z] = dy\sqrt{(1+pp)} + \frac{y\,p\,dp}{\sqrt{(1+pp)}}$: unde erit $[M] = 0$; $[N] = \sqrt{(1+pp)}$; $[P] = \frac{yp}{\sqrt{(1+pp)}}$; reliqui valores $[Q], [R], [S]$, &c. omnes erunt $= 0$. Quocirca pro curva quæſita iſta habebitur æquatio: $0 = (H - \int L\,dx)\sqrt{(1+pp)} - \frac{1}{dx}\,d.\,\frac{yp}{\sqrt{(1+pp)}}(H - \int L\,dx)$. Ponatur, brevitatis gratia, $H - \int L\,dx = V$; erit $V\,dx\sqrt{(1+pp)} = d.\,\frac{ypV}{\sqrt{(1+pp)}} = \frac{Vpp\,dx}{\sqrt{(1+pp)}} + \frac{Vy\,dp}{(1+pp)^{3:2}} + \frac{y\,p\,dV}{\sqrt{(1+pp)}}$, ſeu $V\,dx = \frac{Vy\,dp}{1+pp} + yp\,dV = \frac{Vy\,dp}{1+pp} - yp\,L\,dx$, ob $dV = -L\,dx$. Ponamus eſſe $Z = \Pi$, ita ut maximum eſſe debeat $\int dx \int y\,dx\sqrt{(1+pp)}$, erit $L = 1$ & $\int L\,dx = x$, atque $V = a - x$, ob $H = a$. Erit $(a-x)\,dx = \frac{(a-x)y\,dp}{1+pp} - yp\,dx$. Sit $a - x = u$; erit $dx = -du$, & $dy = -p\,du$, atque habebitur iſta æquatio; $0 = u\,du - y\,dy + \frac{uy\,dp}{1+pp}$, ſeu $u\,du - y\,dy - \frac{uy\,du\,ddy}{du^2+dy^2} = 0$. Ponatur $u = e^t$ & $y = e^t z$: erit $du = e^t\,dt$, & $ddu = 0 = e^t(ddt + dt^2)$, ſeu $ddt = -dt^2$; porro $dy = e^t\,(dz + z\,dt)$ & $ddy = e^t(ddz + 2\,dt\,dz)$; quibus ſubſtitutis, oritur

$dt - zdz - zzdt = \frac{zdt(ddz + 2dtdz)}{dt^2 + (dz + zdt)^2}$. Sit porro $dt = sdz$, erit $ddt = -s^2 dz^2 = sddz + dsdz$, hincque $ddz = -sdz^2 - \frac{dsdz}{s}$. Habebitur ergo hæc æquatio, $sdz - zdz - szzdz = \frac{zs^2dz - zds}{ss + (1 + sz)^2}$; quæ quidem eſt differentialis primi gradus inter duas variabiles s & z tantum; verumtamen ultra integrationem non admittit. Multo minus igitur quicquam effici poterit, ſi in genere quæſtionem conſideremus.

SCHOLION III.

18. Hujus exempli caſus, quo curvam inveſtigavimus, in qua maximum minimumve ſit $\int dx \int y dx \sqrt{(1 + pp)}$, etſi ineſt duplex ſignum integrale, tamen etiam per methodum præcedentis Capitis poteſt reſolvi; id quod ideo operæ pretium eſt oſtendere, ut conſenſus utriuſque methodi declaretur. Præcipue autem hoc opere nova via patefiet reſolvendi plurima alia Problemata circa maxima & minima, quæ adhuc, quantum conſtat, non eſt tacta. Quæſtio ſcilicet eſt, ut pro data abſciſſa AZ $= a$, fiat maximum minimumve hæc expreſſio $\int dx \int y dx \sqrt{(1 + pp)}$, quæ tranſmutatur in hanc $x \int y dx \sqrt{(1 + pp)} - \int xy dx \sqrt{(1 + pp)}$. Ut hæc forma reddatur maximum minimumve, oportet ut ejus valor, pro abſciſſa AZ $= a$, idem ſit pro ipſa curva quæſita az & pro eadem puncto n in ν tranſlato. Ponamus ergo fieri $\int y dx \sqrt{(1 + pp)} = A$, ſi ponatur $x = a$, atque eodem caſu $\int xy dx \sqrt{(1 + pp)} = B$. Jam, elementis mno in mνo tranſmutatis, valor A augebitur ſuo valore differentiali, qui, per Caput præcedens, eſt $= \text{n}\nu . dx(\sqrt{(1 + pp)} - \frac{1}{dx} d. \frac{yp}{\sqrt{(1 + pp)}})$; per eadem præcepta autem quantitatis B valor differentialis prodit $= \text{n}\nu . dx(x\sqrt{(1 + pp)} - \frac{1}{dx} d. \frac{xyp}{\sqrt{(1 + pp)}})$. Quamobrem formulæ propoſitæ $\int dx \int y dx \sqrt{(1 + pp)}$, tranſlato puncto n in ν, pro abſciſſa AZ $= a$,

$= a$, valor erit $= a\left(A + m\,dx\left(\sqrt{(1+pp)} - d.\frac{yp}{\sqrt{(1+pp)}}\right)\right.$ $\left. - B - n\nu\left(x\,dx\sqrt{(1+pp)} - d.\frac{xyp}{\sqrt{(1+pp)}}\right)\right)$, qui æqualis eſſe debet ejuſdem formulæ valori naturali pro abſciſſa $= a$, non mutato puncto n, qui eſt $aA - B$. Hinc proveniet iſta æquatio $(a-x)\,dx\sqrt{(1+pp)} - d.\frac{(a-x)\,yp}{\sqrt{(1+pp)}} = 0$; quæ omnino congruit cum æquatione in ſolutione Exempli inventa.

PROPOSITIO III. PROBLEMA.

19. *Exiſtente* П *functione integrali indeterminata* $\int[Z]\,dx$, *ita ut ſit* $d[Z] = [M]\,dx + [N]\,dy + [P]\,dp + [Q]\,dq + [R]\,dr +$ &c. *ſit* Z *functio quæcunque cum hujus quantitatis* П, *tum quantitatum determinatarum* x, y, p, q, r, s, &c. *ita ut ſit* $dZ = L\,d\Pi + M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr +$ &c. *invenire curvam* az, *quæ, pro data abſciſſa* $AZ = a$, *habeat maximum minimumve valorem formulæ* $\int Z\,dx$.

SOLUTIO.

Augmentum $n\nu$, quod uni applicatæ Nn accedere concipitur, ita remotum a prima applicata Hh capiatur, ut nullam mutationem inferat in valorem formulæ $\int Z dx$ abſciſſæ AH reſpondentem, atque tantum hujus formulæ valores ſequentibus poſt H abſciſſæ elementis reſpondentes mutationes patiantur, qui ſunt $Z\,dx$, $Z'\,dx$, $Z''\,dx$, $Z'''\,dx$, &c. uſque ad ultimum abſciſſæ elementum in Z. Horum igitur valorum incrementa a tranſlatione puncti n in ν orta, ſi in unam ſummam conjiciantur, & nihilo æquales ponantur, dabunt æquationem pro curva quæſita. Incrementa autem horum valorum obtinebuntur eos differentiando, & loco differentialium eos valores ſcribendo, quos ſupra, tam in ultima Propoſitione præcedentis Capitis quam prima hujus, ex tranſlatione n in ν oriri invenimus: ita erit

$$d.\ Z\,dx = dx\,(L\,d\pi + M dx + N dy + P dp + \&c.\ )$$
$$d.\ Z'dx = dx\,(L'd\pi' + M'dx + N'dy' + P'dp' + \&c.\ )$$
$$d.\ Z''dx = dx\,(L''d\pi'' + M''dx + N''dy'' + P''dp'' + \&c.\ )$$
&c.

Quod ſi nunc loco differentialium $d\pi$, $d\pi'$, $d\pi''$ &c. dy, dy', dy'', &c. dp, dp', dp'', &c. dq, dq', dq'', &c. valores ſupra inventi ſubſtituantur, & eodem modo, quo ante uſi ſumus, in unam ſummam conferantur, prodibit formulæ $\int Z dx$ pro abſciſſa $AZ = a$ valor differentialis =

$$n\nu.\, dx\,(\,[N]\,(H - \int L dx) - \frac{d.[P](H - \int L dx)}{dx} + \frac{dd.[Q](H - \int L dx)}{dx^2}$$
$$- \frac{d^3.[R](H - \int L dx)}{dx^3} + \frac{d^4.[S](H - \int L dx)}{dx^4} - \&c.\ )$$
$$+ n\nu.\ dx\,(\,N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \&c.).$$

Atque ex hoc reſultabit æquatio pro curva quæſita hæc :

$$0 = [N]\,(H - \int L dx) - \frac{d.[P]\,(H - \int L dx)}{dx}$$
$$+ \frac{dd.[Q]\,(H - \int L dx)}{dx^2} - \frac{d^3.[R]\,(H - \int L dx)}{dx^3} + \&c.$$
$$+ N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \&c.$$ ubi notandum eſſe H valorem formulæ $\int L dx$, qui oritur poſito $x = a$. Q. E. I.

COROLL. I.

20. Regula igitur Capite præcedente inventa amplior eſt reddita; nunc enim curvam definire poſſumus, maximum minimumve habentem valorem formulæ $\int Z dx$, ſi Z non ſolum eſt functio quantitatum determinatarum x, y, p, q, r, &c. ſed etiam unam quantitatem integralem indefinitam $\int [Z] dx$ in ſe complectitur: dummodo $[Z]$ ſit functio determinata.

Co-

Coroll. II.

21. Quin etiam ſi plures hujuſmodi quantitates integrales indefinitæ fuerint in Z; ſolutio uſurpari poterit. Nam qualis expreſſio ex una ejuſmodi formula indefinita in valorem differentialem eſt ingreſſa, tales ex ſingulis, ſi plures affuerint, naſcentur & ad valorem differentialem accedent.

Coroll. III.

22. Quoniam Z hic ponitur functio non ſolum quantitatum definitarum x, y, p, q, r &c. ſed etiam quantitatis indefinitæ $\Pi = \int [Z]\, dx$, ob $dZ = L\, d\Pi + M\, dx + N\, dy + P\, dp + Q\, dq +$ &c. etiam quantitates M, N, P, Q &c. hanc formulam integralem $\Pi = \int [Z]\, dx$ involvent; atque etiam ipſa quantitas L, niſi forte Π in Z unicam habeat dimenſionem.

Coroll. IV.

23. Hanc ob rem, in æquatione pro curva inventa, inerunt quantitates integrales duplicis generis, ſcilicet $\int L\, dx$, atque $\int [Z]\, dx$: ex quo, ſi æquatio inventa per differentiationem ab his formulis liberari debeat, ad multo altiorem differentialium gradum aſſurget, quam quidem ipſa forma oſtendit.

Coroll. V.

24. Pervenietur autem, eliminando has formulas integrales, ad æquationem differentialem duobus gradibus altiorem. Quod ſi enim æquatio reſultans, ſi evolvatur, ſit differentialis n gradus; tum primo ex ea definiatur valor formulæ $\int L\, dx$, & differentiatione inſtituta, devenietur ad æquationem differentialem $n + 1$ graduum, in qua adhuc inerit formula $\int [Z]\, dx$, quæ ulterius reducta, & a formula $\int [Z]\, dx$ per differentiationem liberata, fiet differentialis gradus $n + 2$.

SCHOLION I.

25. Etſi autem numerus punctorum, per quæ curva quæſita tranſire debet, a gradu *differentialitatis* pendet, tamen hoc caſu non per numerum $n+2$ definiri poteſt. Æquatio enim hæc differentialis $n+2$ graduum, poteſtate quidem involvit $n+2$ conſtantes, verum eæ non omnes ſunt arbitrariæ. Una namque conſtans ex eo determinatur, quod integrale $\int[Z]\,dx$ obtinere debeat valorem, non vagum, ſed talem qualem in quantitate Z obtinet, hoc eſt, qui evaneſcat poſito $x=0$, ſiquidem hæc conditio fuerit in formula $\int Z\,dx$ aſſumta. Deinde pari modo una conſtans definitur formula $\int L\,dx$, quæ, uti poſuimus, evaneſcere debet poſito $x=0$. Quocirca tantum n ſupererunt conſtantes mere arbitrariæ, quæ totidem præbebunt puncta, quibus Problema determinabitur. Similiter igitur, uti in præcedente Capite, Problema, ut ſit determinatum, ita erit proponendum, ut inter omnes curvas per data n puncta tranſeuntes ea determinetur, quæ pro data abſciſſa $x=a$ contineat valorem formulæ $\int Z\,dx$ maximum minimumve. Ad hanc igitur dijudicationem inſtituendam, æquatio inventa debebit evolvi; hoc eſt, omnes differentiationes indicatæ actu perfici debebunt; quo facto, patebit quanti gradus differentialia inſint, ex hocque gradu habebitur numerus n. Quantum autem inſuper circa hunc numerum n obſervare liceat, in Exemplis ſequentibus videbimus.

EXEMPLUM I.

26. *Invenire curvam, quæ, pro data abſciſſa* $AZ=a$, *habeat valorem formulæ* $\int yx\,dx\int y\,dx$ *maximum vel minimum, integrali* $\int y\,dx$ *ita accipiendo, ut evaneſcat poſito* $x=0$.

Erit igitur $\Pi=\int y\,dx$, & $[Z]=y$; unde fiet $[N]=1$, reliquis litteris $[M]$, $[P]$, $[Q]$, &c. exiſtentibus $=0$. Porro erit $Z=yx\Pi$ & $dZ=yx\,d\Pi+y\Pi\,dx+x\Pi\,dy$; ex quo habebitur $L=yx$; $M=y\Pi$ & $N=x\Pi$, $P=Q=R$, &c. $=0$.

$=0$. Ex his formabitur pro curva quæſita iſta æquatio; $0 = (H - \int yxdx) + x\Pi$ ſeu $\int yxdx = H + x\int ydx$, ubi H eſt valor formulæ $\int yxdx$, qui prodit poſito $x = a$. Perſpicuum autem eſt hinc nullam pro aliqua linea curva æquationem oriri: differentiatione enim inſtituta, fit $dx\int ydx = 0$, porroque $y = 0$, quæ eſt æquatio pro linea recta in axem A Z incidente.

EXEMPLUM II.

27. *Invenire curvam, quæ, pro data abſciſſa* $AZ = a$, *habeat valorem formulæ* $\int ydx\int dx\sqrt{(1+pp)}$ *maximum vel minimum.*

Quoniam igitur eſt $\Pi = \int dx\sqrt{(1+pp)}$, erit $[Z] = \sqrt{(1+pp)}$ & $[P] = \frac{p}{\sqrt{(1+pp)}}$: Porro erit $Z = y\Pi$ & $L = y$; & $N = \Pi$; reliquæ litteræ omnes evaneſcunt. Hinc ergo reſultabit iſta æquatio pro curva quæſita: $0 = \frac{1}{dx} \times d.\frac{p(H - \int ydx)}{\sqrt{(1+pp)}} + \Pi$ ſeu $\Pi dx = d.\frac{(H - \int ydx)p}{\sqrt{(1+pp)}} = \frac{(H - \int ydx)dp}{(1+pp)^{\frac{3}{2}}} - \frac{ypdx}{\sqrt{(1+pp)}}$; ergo $dx\int dx\sqrt{(1+pp)} = \frac{(H - \int ydx)dp}{(1+pp)^{\frac{3}{2}}} - \frac{ypdx}{\sqrt{(1+pp)}}$. Quia igitur fit $\int ydx = H$, poſito $x = a$, eodem caſu fiet $\int dx\sqrt{(1+pp)} = \frac{-yp}{\sqrt{(1+pp)}} =$ arcui curvæ abſciſſæ a reſpondenti. Quæ conditio adimpleri debet per determinationem unius conſtantis, quæ per integrationem ingredietur. Eſt autem actu hæc æquatio differentialis ſecundi gradus, quæ vero bis debet differentiari, antequam a formulis integralibus $\int ydx$ & $\int dx\sqrt{(1+pp)}$ liberetur: hocque modo ad gradum ſextum aſſurget, & poteſtate ſex conſtantes involvet; quarum duæ inde determinabuntur, quod facto $x = 0$ evaneſcere debent formulæ $\int ydx$ & $\int dx\sqrt{(1+pp)}$. Ipſa autem æquatio ita fiet intricata, ut ejus tractatio ſuſcipi non mereatur.

EXEMPLUM III.

28. *Invenire curvam, in qua pro data abſciſſa ſit* $\int\frac{dx}{p}\int y\,dx$ *maximum vel minimum.*

Hic erit $\Pi = \int y\,dx$, & $[Z] = y$ & $[N] = 1$; deinde cum ſit $Z = \frac{\Pi}{p}$, erit $L = \frac{1}{p}$ & $P = -\frac{\Pi}{pp}$; reliquæ litteræ omnes evaneſcunt. Hinc ergo prodit iſta æquatio. $0 = H - \int\frac{dx}{p} + \frac{1}{dx}d.\frac{\Pi}{pp}$; ſeu $0 = H - \int\frac{dx}{p} + \frac{y}{pp} - \frac{2\Pi\,dp}{p^3\,dx}$ Poſito ergo $x = a$, quo caſu fit $\int\frac{dx}{p} = H$; erit $y\,dx = \frac{2\Pi\,dp}{p}$. Differentietur ea æquatio, eritque $0 = -\frac{dx}{p} + \frac{dx}{p} - \frac{2y\,dp}{p^2} - \frac{2y\,dp}{p^3} + \frac{6\Pi\,dp^2}{p^4\,dx} - \frac{2\Pi\,ddp}{p^3\,dx}$. Seu $0 = 3\Pi\,dp^2 - 2yp\,dx\,dp - \Pi p\,ddp$; quæ æquatio commode fit integrabilis, ſi dividatur per $\Pi p\,dp$, prodit enim $0 = \frac{3\,dp}{p} - \frac{2y\,dx}{\Pi} - \frac{ddp}{dp}$, cujus integrale eſt $C = 3\,lp = 2\,l\Pi - l\,\frac{dp}{dx}$. Seu $C\Pi^2\,dp = p^3\,dx$; poſito ergo $x = a$, cum eſſe debeat $y\,dx = \frac{2\Pi\,dp}{p}$; erit ex hac æquatione $C\Pi y = 2p^2$, qua una conſtans definietur. Erit ergo $\Pi = \sqrt{\frac{p^3\,dx}{C\,dp}} = \frac{2yp\,dx\,dp}{3\,dp^2 - p\,ddp}$, ſeu $3\,dp^2 - p\,ddp = \frac{2y\,dp\sqrt{dx\,dp}}{b\sqrt{bp}}$, quæ æquatio eſt differentialis tertii gradus, & propterea præter conſtantem b (poſuimus autem $\frac{1}{b^3}$ loco C), tres novas conſtantes involvit. Harum una determinabitur, eo quod, poſito $x = a$, fieri debeat $\frac{\Pi y}{b^3} = 2pp$; alia vero inde quod, poſito $x = 0$, eſſe debeat $\Pi = 0$, ſeu $\frac{p^3\,dx}{dp} = 0$. Reliquæ

liquæ binæ conſtantes manent arbitrariæ, ac propterea curva quæſita per duo data puncta per quæ tranſeat, debet determinari.

EXEMPLUM IV.

29. *Invenire curvam* az *ad abſciſſam* $AZ = a$ *relatam, in qua ſit* $\int dx \frac{\int yx\,dx}{\int y\,dx}$ *maximum vel minimum.*

Hoc exemplum ideo afferre viſum eſt, ut appareat quomodo quæſtiones ejuſmodi ſint reſolvendæ, ſi duæ pluresve formulæ integrales indefinitæ adſint. Sit igitur $\int yx\,dx = \Pi$ & $\int y\,dx = \pi$: & poſito $d\Pi = [Z]\,dx$, & $d\pi = [z]\,dx$, erit $[Z] = yx$, & $[z] = y$. Quod ſi nunc littera minuſcula $[z]$ ſimili modo tractetur quo majuſcula $[Z]$, ita ut ſit $d[z] = [m]\,dx + [n]\,dy + [p]\,dp +$ &c. erit $[M] = y$ & $[N] = x$, itemque $[n] = 1$. Deinde cum ſit $Z = \frac{\Pi}{\pi}$, erit $dZ = \frac{d\Pi}{\pi} - \frac{\Pi\,d\pi}{\pi^2}$. Ponatur $\frac{1}{\pi} = L$ & $\frac{\Pi}{\pi^2} = l$; atque habebitur ob N & P, Q, R, &c. $= 0$, iſta pro curva quæſita æquatio, $0 = x\left(H - \int \frac{dx}{\pi}\right) - 1\left(h - \int \frac{\Pi\,dx}{\pi^2}\right)$, ubi ſit $\int \frac{dx}{\pi} = H$ & $\int \frac{\Pi\,dx}{\pi^2} = h$, ſi ponatur $x = a$. Cum igitur ſit $Hx - x\int \frac{dx}{\pi} = h - \int \frac{\Pi\,dx}{\pi^2}$ erit differentiando $H - \int \frac{dx}{\pi} - \frac{x}{\pi} = -\frac{\Pi}{\pi^2}$. Poſito ergo $x = a$, fieri debet $\Pi = \pi x$. Differentietur denuo, prodibitque $-\frac{2}{\pi} + \frac{xy}{\pi^2} = -\frac{yx}{\pi^2} + \frac{2\Pi y}{\pi^3}$, ſeu $xy - \pi = \frac{\Pi y}{x}$; hincque $\Pi = \pi x - \frac{\pi\pi}{y}$. Si porro differentiatio inſtituatur, habebitur $yx\,dx = \pi\,dx + yx\,dx - 2\pi\,dx + \frac{\pi\pi\,dy}{yy}$, ſeu $yy\,dx = \pi\,dy$, vel $\frac{y\,dx}{\pi} = \frac{dy}{y}$. Quoniam vero, poſito

poſito $x = 0$, fit $\pi = 0$, fiet hoc caſu $\frac{yy\,dx}{dy} = 0$. Æquatio autem $\frac{y\,dx}{\pi} = \frac{dy}{y}$, ob $y\,dx = d\pi$, integrata dat $\pi = by$; ideoque facto $x = 0$ evaneſcere debet y. Ex æquatione $\pi = by$ autem ſequitur $y\,dx = b\,dy$; hincque $x = bly - bl0$, ſiquidem $\pi = by$ evaneſcere debeat, poſito $x = 0$; quo caſu fieret $y = 0$, & curva abiret in rectam in axem AZ incidentem. Sin autem ponamus, poſito $x = 0$ valorem $\pi = \int y\,dx$ non evaneſcere oportere, ſed fieri $= bc$, erit $x = bl\frac{y}{c}$, quæ eſt æquatio pro Curva logarithmica. Ad hanc penitus determinandam, quæratur valor $\Pi = \int yx\,dx$; quia eſt $y\,dx = b\,dy$, erit $yx\,dx = bx\,dy$, & $\Pi = bxy - b\pi + Conſt.$ ſeu $\Pi = bbyl\frac{y}{c} - bby + C$. Oporteat autem Π eſſe $= 0$; poſito $x = 0$, ſeu $y = c$, erit $\Pi = bbyl\frac{y}{c} + bb(c - y)$. Jam ponatur $x = a$, erit $l\frac{y}{c} = \frac{a}{b}$, & $y = ce^{a:b}$: hoc vero caſu, neceſſe eſt ut ſit $\Pi = \pi x$, ſeu $abce^{a:b} + bbc - bbce^{a:b} = abce^{a:b}$, hincque $e^{a:b} = 1$, unde erit, vel $a = 0$, vel $b = \infty$. Incommodum hoc inde oritur, quod poſuimus fieri $\Pi = 0$, facto $x = 0$. Ponamus igitur, poſito $y = g$, tum eo caſu Π evaneſcere, erit $\Pi = bbyl\frac{y}{c} - bby + bbg - bbgl\frac{g}{c}$. Jam poſito $x = a$, quo caſu fieri debet $\Pi = \pi x = a\pi$; erit $abce^{a:b} - bbce^{a:b} + bbg - bbgl\frac{g}{c} = abce^{a:b}$, hincque $e^{a:b} = \frac{g}{c}(1 - l\frac{g}{c})$, ſeu $b = \frac{a}{l\frac{g}{c}(1 - l\frac{g}{c})}$ ideoque $x = \frac{a(ly - lc)}{lg(1 - l\frac{g}{c}) - lc}$. Quæ eſt æquatio curvam

penitus

Digitized by Google

penitus determinans, ita ut nullum curvæ punctum pro arbitrio accipi liceat.

SCHOLION II.

30. Per hoc igitur Problema, non ſolum illæ quæſtiones curvam pro data abſciſſa maximum minimumve habentem formulam $\int Z dx$ deſiderantes reſolvi poſſunt, in quibus Z præter quantitates determinatas x, y, p, q, r, s, &c. unam formulam integralem $\Pi = \int [Z] dx$ complectitur; ſed etiamſi plures ejuſmodi formulæ affuerint. Interim tamen notandum eſt has formulas integrales $\Pi = \int [Z] dx$ in functione Z contentas, ita comparatas eſſe debere, ut $[Z]$ ſit functio determinata, hoc eſt functio quantitatum x, y, p, q, r, &c. nullas ultra formulas integrales involvens. Hanc ob rem, nunc inveſtigemus methodum reſolvendi ejuſmodi Problemata, quando iſta functio $[Z]$ non eſt determinata, ſed præter x, y, p, q, &c. formulam integralem novam $\pi = \int [z] dx$ involvit. Ne autem ſolutio nimium fiat prolixa, non ultra differentialia ſecundi gradus conſiderabimus. Jam enim intelligitur ſi ſolutio fuerit adornata uſque ad differentialia ſecundi gradus, tum per inductionem, ſolutionem ad quoſque ulteriores gradus extendi poſſe. Hunc in finem nobis erit Ll prima applicata deſignanda per y, a qua tertia quæ ſequitur $Nn = y''$ particula $n\nu$ augeri concipiatur: Ex hoc augmento naſcentur ſequentia quantitatum y, p, & q, cum ſuis derivativis incrementa

$d.\,y = 0$	$d.\,p = 0$	$d.\,q = +\frac{n\nu}{dx^2}$
$d.\,y' = 0$	$d.\,p' = +\frac{n\nu}{dx}$	$d.\,q' = +\frac{2n\nu}{dx^2}$
$d.\,y'' = +n\nu$	$d.\,p'' = -\frac{n\nu}{dx}$	$d.\,q'' = +\frac{n\nu}{dx^2}$

quæ Tabella ſufficiet ad Problemata quæcunque reſolvenda, uti ex ſequente Propoſitione intelligetur.

PROPOSITIO IV. PROBLEMA.

31. *Sit* $\pi = \int [z] dx$ *&* $d[z] = [m]dx + [n]dy + [p]dp + [q]dq$, *atque quantitas* $[Z]$ *ita involvat formulam integralem* π, *ut ſit* $d[Z] = Zd\pi + [M]dx + [N]dy + [P]dp + [Q]dq$. *Jam poſito* $\Pi = \int [Z] dx$, *ſit* Z *functio ipſarum* x, y, p, q, *itemque ipſius* Π, *ita ut ſit* $dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq$. *His poſitis, oporteat definiri curvam* az, *quæ pro data abſciſſa* $AZ = a$, *habeat valorem formulæ* $\int Z dx$ *maximum vel minimum.*

SOLUTIO.

Ut in Scholio præcedente monuimus, eſt nobis abſciſſa AL $= x$, & applicata $Ll = y$; abſciſſæ autem $AL = x$ reſpondeat valor $\int Z dx$ qui a particula $n\nu$ non afficietur. Ex quo valor differentialis ex ſequentibus abſciſſæ elementis determinari debebit, quibus reſpondebunt valores Zdx, $Z'dx$, $Z''dx$, $Z'''dx$, $Z^{IV}dx$, &c. uſque ad ultimum abſciſſæ totius propoſitæ AZ elementum in Z. Invenientur autem ſingulorum horum terminorum valores differentiales per differentiationem; ſubſtituendo loco differentialium dy, dp, dq, valores paragrapho præcedenti indicatos. Erit igitur

$$d.Zdx = dx\left(Ld\Pi + \frac{Q.n\nu}{dx^2}\right)$$
$$d.Z'dx = dx\left(L'd\Pi' + \frac{P'n\nu}{dx} - \frac{2Q'.n\nu}{dx^2}\right)$$
$$d.Z''dx = dx\left(L''d\Pi'' + N''.n\nu - \frac{P''.n\nu}{dx} + \frac{Q''.n\nu}{dx^2}\right)$$
$$d.Z'''dx = dx.L'''d\Pi'''$$
$$d.Z^{IV}dx = dx.L^{IV}d\Pi^{IV}$$
&c.

Supereſt igitur ut per $n\nu$ definiamus differentialia $d\Pi$, $d\Pi'$, $d\Pi''$, $d\Pi'''$ &c. hoc eſt valores differentiales quantitatum Π, Π', Π'', Π''' &c. Eſt vero $\Pi =$

$$\begin{aligned}
\Pi &= \int [Z]dx \\
\Pi' &= \int [Z]dx + [Z]dx \\
\Pi'' &= \int [Z]dx + [Z]dx + [Z']dx \\
\Pi''' &= \int [Z]dx + [Z]dx + [Z']dx + [Z'']dx \\
\Pi^{IV} &= \int [Z]dx + [Z]dx + [Z']dx + [Z'']dx + [Z''']dx
\end{aligned}$$

&c.

Ubi notandum eſt quantitatis $\int [Z]dx$ valorem differentialem eſſe $= o$, eo quod particula $n\nu$ nullam mutationem infert in abſciſſam A L ad quam $\int [Z]dx$ refertur. Tantum igitur terminorum differentialium $[Z]dx$, $[Z']dx$, $[Z'']dx$ &c. valores differentiales inveſtigari oportebit. Erit autem.

$$\begin{aligned}
d.[Z]dx &= dx\left([L]d\pi + \frac{[Q.]n\nu}{dx^2}\right) \\
d.[Z']dx &= dx\left([L']d\pi' + \frac{[P']n\nu}{dx} - \frac{2[Q'.]n\nu}{dx^2}\right) \\
d.[Z'']dx &= dx\left([L'']d\pi'' + [N''.]n\nu - \frac{[P''.]n\nu}{dx} + \frac{[Q''.]n\nu}{dx^2}\right) \\
d.[Z''']dx &= dx[L''']d\pi''' \\
d.[Z^{IV}]dx &= dx.[L^{IV}]d\pi^{IV}
\end{aligned}$$

&c.

Nunc porro definiendi ſunt valores differentiales quantitatum π, π', π'', π''', &c. per $n\nu$, quos loco $d\pi$, $d\pi'$, $d\pi''$, &c. ſubſtitui oportet. Cum autem ſit $\pi = \int [z]dx$, & in $[z]$ differentialia ſecundum gradum ſuperantia non ineſſe ponantur, fiet valor differentialis ipſius π, ſeu $d\pi = o$, ad ſequentium autem quantitatum π', π'', π''' &c. valores differentiales inveniendos, notaſſe conveniet eſſe

$$\begin{aligned}
\pi &= \int [z]dx \\
\pi' &= \int [z]dx + [z]dx \\
\pi'' &= \int [z]dx + [z]dx + [z']dx \\
\pi''' &= \int [z]dx + [z]dx + [z']dx + [z'']dx \\
\pi^{IV} &= \int [z]dx + [z]dx + [z']dx + [z'']dx + [z''']dx
\end{aligned}$$

&c.

 Erit

Erit autem

$$d.[z]dx = nv.\,dx\,\frac{[q]}{dx^2}$$
$$d.[z']dx = nv.\,dx\left(\frac{[p']}{dx} - \frac{2[q']}{dx^2}\right)$$
$$d.[z'']dx = nv.\,dx\left([n''] - \frac{[p'']}{dx} + \frac{[q'']}{dx^2}\right)$$
$$d.[z''']dx = 0$$
$$d.[z^{IV}]dx = 0$$
&c.

Ex his itaque obtinebitur

$$d.\pi = 0$$
$$d.\pi' = nv.\,dx.\frac{[q]}{dx^2}$$
$$d.\pi'' = nv.\,dx\left(\frac{[p']}{dx} - \frac{[q]}{dx^2} - \frac{2d[q]}{dx^2}\right)$$
$$d.\pi''' = nv.\,dx\left([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\right)$$
$$d.\pi^{IV} = nv.\,dx\left([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\right)$$
$$d.\pi^{V} = nv.\,dx\left([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\right)$$

omneſque ſequentes valores inter ſe erunt æquales. Quod ſi jam hi valores inventi ſubſtituantur, erit

$$d.[Z]dx = nv.\,dx.\frac{[Q]}{dx^2}$$
$$d.[Z']dx = nv.\,dx\left(\frac{[L'][q]}{dx} + \frac{[P']}{dx} - \frac{2[Q]}{dx^2}\right)$$
$$d.[Z'']dx = nv.\,dx\left([L'']dx\left(\frac{[p']}{dx} - \frac{[q] - 2d[q]}{dx^2}\right) + [N'] - \frac{[P'']}{dx} + \frac{[Q'']}{dx^2}\right)$$

d.

$$d.[Z'''] dx = nv.dx.[L'''] dx ([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2})$$
$$d.[Z^{IV}] dx = nv.dx.[L^{IV}] dx ([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2})$$
$$d.[Z^{V}] dx = nv.dx [L^{V}] dx ([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2})$$
&c.

Hinc porro deducitur:

$$d\,\Pi = 0$$
$$d.\Pi' = nv.dx.\frac{[Q]}{dx^2}$$
$$d.\Pi'' = nv.dx([L']dx.\frac{[q]}{dx^2} + \frac{[P']}{dx} - \frac{[Q] + 2d[Q]}{dx^2})$$
$$d.\Pi''' = nv.dx\ ([L''][p'] - \frac{[q]d[L'] + 2[L']d[q]}{dx}$$
$$+ [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2})$$
$$d.\Pi^{IV} = nv.dx([L'''] dx ([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2})$$
$$+ [L''][p] - \frac{[q]d[L'] + 2[L']d[q]}{dx} + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2})$$
$$d.\Pi^{V} = nv.dx(([L'''] dx + [L^{IV}] dx)([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2})$$
$$+ [L''][p] - \frac{[q]d[L'] + 2[L']d[q]}{dx} + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2})$$
&c.

Ex his jam orientur ſequentes determinationes:

$$d.Z dx = nv.dx.\frac{Q}{dx^2}$$
$$d.Z' dx = nv.dx(L'dx.\frac{[Q]}{dx^2} + \frac{P'}{dx} - \frac{2Q'}{dx^2})$$
$$d.Z'' dx = nv.dx(L''dx([L']dx.\frac{[q]}{dx^2} + \frac{[P']}{dx} - \frac{[Q] + 2d[Q]}{dx^2})$$
$$+ N'' - \frac{P''}{dx} + \frac{Q''}{dx^2})$$

 d.

Digitized by Google

$$d.\,Z''dx = nv.\,dx.\,L'''dx\Big([L''][p'] - \frac{[q]d[L'] + 2[L']d[q]}{dx} + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2}\Big)$$

$$d.\,Z^{IV}dx = nv.\,dx\,L^{IV}dx\Big([L''']dx\Big([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\Big) + [L''][p'] - \frac{[q]d[L] + 2[L]d[q]}{dx} + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2}\Big)$$

$$d.Z^{V}dx = nv.dx.L^{V}dx\Big(([L''']dx + [L^{IV}]dx)\Big([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\Big) + [L''][p'] - \frac{[q]d[L'] + 2[L']d[q]}{dx} + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2}\Big)$$

$$d.Z^{VI}dx = nv.dx.L^{VI}dx\Big(([L''']dx + [L^{IV}]dx + [L^{V}]dx)\Big([n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2}\Big) + [L''][p'] - \frac{[q]d[L'] + 2[L']d[q]}{dx} + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2}\Big)$$

&c.

Ut hi valores omnes eo commodius ad ſe invicem addi queant; ponamus brevitatis gratia $[b] = [n''] - \frac{d[p']}{dx} + \frac{dd[q]}{dx^2} = [n] - \frac{d[p]}{dx} + \frac{dd[q]}{dx^2}$; & $[H] = [L][p] - \frac{[q]d[L] + 2[L]d[q]}{dx} + [N] - \frac{d[P]}{dx} + \frac{dd[Q]}{dx^2}$: eritque ſumma omnium, hoc eſt valor differentialis formulæ propoſitæ $\int Z\,dx$, ut ſequitur.

$$nv.\,dx\Big(N - \frac{dP}{dx} + \frac{ddQ}{dx^2}\Big) + nv.\,dx\Big(L[P] - \frac{[Q]dL - 2Ld[Q]}{dx}\Big)$$
$$+ nv.\,dx.\,L[L][q] + nv.\,dx.\,[H](L'''dx + L^{IV}dx + L^{V}dx + \&c.\ in\ Z) + nv.\,dx.\,[b]\big(L^{IV}dx.\,[L''']dx + L^{V}dx([L''']dx + [L^{IV}]dx) + L^{VI}dx([L''']dx + [L^{IV}]dx + [L^{V}]dx) + L^{VII}dx([L''']dx + [L^{IV}]dx + [L^{V}]dx + [L^{VI}]dx) + \&c.\big)$$

Binæ igitur hic habentur ſeries infinitæ, a termino Ll uſque ad Zz progredientes, quarum illius $L'''dx + L^{IV}dx + L^{V}dx + \&c.$ ſumma exprimi poteſt per $H - \int L\,dx$, denotante H valorem ipſius $\int L\,dx$, poſito $x = a$. Quo autem valorem alterius ſeriei inveſtigemus, ponatur ejus ſumma $= S$, ita ut ſit $S = L^{IV}dx.$

$[L''']$

$[L''']\,dx + L^{v}\,dx\,([L''']\,dx + [L^{iv}]\,dx) +$ &c. Sumatur valor ſequens $S' = S + dS$, erit $S + dS = L^{v}\,dx.\,[L^{iv}]\,dx + L^{vi}\,dx\,([L^{iv}]\,dx + [L^{v}]\,dx) +$ &c. qui ab illo ſubtractus relinquet, $-\,dS = L^{iv}\,[L''']\,dx^2 + L^{v}\,[L''']\,dx^2 + L^{vi}\,[L''']\,dx^2 +$ &c. ſeu $-\,dS = [L''']\,dx\,(L^{iv}\,dx + L^{v}\,dx + L^{vi}\,dx +$ &c.) ideoque $-\,dS = [L''']\,dx\,(H - \int L\,dx)$, & integrando $S = G - \int [L]\,dx\,(H - \int L\,dx)$, conſtante G ita aſſumta, ut fiat $S = 0$ ſi ponatur $x = a$. His inventis fiet valor differentialis formulæ propoſitæ $\int Z\,dx = nv.\,dx\,(N$

$$- \frac{dP}{dx} + \frac{ddQ}{dx^2} + L[P] - \frac{[Q]\,dL + 2L\,d[Q]}{dx} + L[L][q]$$

$$+ (H - \int L\,dx)\,([L][p] - \frac{[q]\,d[L] + 2[L]\,d[q]}{dx}$$

$$+ [N] - \frac{d[P]}{dx} + \frac{dd[Q]}{dx^2}) + (G - \int [L]\,dx\,(H - \int L\,dx))$$

$([n] - \frac{d[p]}{dx} + \frac{dd[q]}{dx^2}))$. Hæc expreſſio autem in ſequentem formam tranſmutari poteſt, ex qua facilius valor differentialis formari poterit, ſi differentialia altiorum graduum quam ſecundi, tam in Z quam in $[Z]$ & $[z]$ inſint. Erit ſcilicet formulæ $\int Z\,dx$ valor differentialis abſciſſæ $AZ = a$ reſpondens

$$= nv.\,dx\,(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{\&c.})$$

$$+ nv.dx\,([N](H - \int L dx - \frac{d.[P](H - \int L dx)}{dx} + \frac{dd.[Q](H - \int L dx)}{dx^2}$$

$$- \frac{d^3.[R](H - \int L dx)}{dx^3} + \frac{d^4.[S](H - \int L dx)}{dx^4} - \text{\&c.}) + nv.\,dx$$

$$([n](G - \int [L] dx (H - \int L dx)) - \frac{d.[p](G - \int [L] dx (H - \int L dx))}{dx}$$

$$+ \frac{dd.[q](G - \int [L] dx (H - \int L dx))}{dx^2} - \frac{d^3[r](G - \int [L] dx (H - \int L dx))}{dx^3}$$

+ &c.). Invento autem valore differentiali, ſi is ponatur $= 0$; habebitur æquatio pro curva quæſita. *Q. E. I.*

COROLL. I.

32. Inventus igitur eſt valor differentialis pro formula $\int Z\,dx$ latius patente, quam quidem in Propoſitione eſt aſſumta: ſcilicet ſi fuerit $dZ = L\,d\Pi + M\,dx + N\,dy + P\,dp + Q\,dq + R\,dR +$ &c. atque exiſtente $d\Pi = [Z]\,dx$, ſi ſit $d[Z] = [L]\,d\pi + [M]\,dx + [N]\,dy + [P]\,dp + [Q]\,dq + [R]\,dr +$ &c. itemque ſi poſito $d\pi = [z]\,dx$ fuerit $d[z] = [m]\,dx + [n]\,dy + [p]\,dp + [q]\,dq + [r]\,dr +$ &c. Quoticunque nimirum gradus differentialia inſint in quantitatibus Z, $[Z]$, & $[z]$ ſolutio data inſerviet.

COROLL. II.

33. Quod ſi ponatur $H - \int L\,dx = T$, & $G - \int [L]\,dx\,(H - \int L\,dx) = V$, erit valor differentialis

$$= n\nu.\,dx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \text{\&c.}\right)$$
$$+ n\nu.\,dx\left([N]\,T - \frac{d.[P]\,T}{dx} + \frac{dd.[Q]\,T}{dx^2} - \frac{d^3\,[R]\,T}{dx^3} + \text{\&c.}\right)$$
$$+ n\nu.\,dx\left([n]\,V - \frac{d.[p]\,V}{dx} + \frac{dd.[q]\,V}{dx^2} - \frac{d^3\,[r]\,V}{dx^3} + \text{\&c.}\right)$$

COROLL. III.

34. Hinc igitur æquatio pro curva quæſita erit hæc, $0 = N + [N]\,T + [n]\,V - \frac{d(P + [P]T + [p]\,V)}{dx} + \frac{dd(Q + [Q]T + [q]\,V)}{dx^2} - \frac{d^3(R + [R]\,T + [r]\,V)}{dx^3} +$ &c. cujus progreſſionis lex, ſi forte opus ſit pluribus terminis, ſponte patet.

COROLL. IV.

35. Quin etiam hinc reſolvi poterunt ejuſmodi Problemata, in quibus Z non unam, ſed plures iſtiuſmodi formulas integrales

les indefinitas Π in ſe complectitur; vel etiam ſi $[Z]$ plures ejuſmodi formulas $\pi = \int [z]\, dx$ in ſe contineat.

COROLL. V.

36. Denique, etſi poſuimus eſſe $[z]$ functionem determinatam, tamen per inductionem hinc modus patet valorem differentialem formandi, ſi ulterius $[z]$ in ſe contineat formulam integralem indefinitam.

SCHOLION.

37. Latiſſime igitur ſolutio hujus Problematis patet, quia non ſolum precedentia Problemata omnia in ſe complectitur, atque ipſi caſui propoſito ſatisfacit, verum etiam per inductionem ad caſus qualeſcunque magis inrricatos accommodari poteſt. Quod ut facilius percipiatur, ponamus in $[z]$ inſuper ineſſe formulam integralem $\pi = \int \zeta\, dx$, ita ut ſit $\zeta = [l]\, d\pi + [m]\, dx + [n]\, dy + [p]\, dp + [q]\, dq +$ &c. exiſtente $d\zeta = \mu\, dx + \nu\, dy + \Phi\, dp + \chi\, dq +$ &c. Jam ad valorem differentialem determinandum, præter quantitates integrales binas T & V, tertia debet definiri W, ita comparata ut ſit $W = F - \int [l]\, dx\, (G - \int [L]\, dx\, (H - \int L\, dx))$ quæ evaneſcat poſito $x = a$. Hocque facto, erit valor differentialis $= n\nu . dx\, (N + [N]\, T + [n]\, V + \nu W - \frac{d(P + [P]\, T + [p]\, V + \Phi W)}{dx} + \frac{dd.(Q + [Q]\, T + [q]\, V + \chi W)}{dx^2} -$ &c.) Quamobrem nequidem maximi minimive formula excogitari poterit, quæ non in ſolutione eſſet contenta, aut ex talibus formulis compoſita, ad quas iſta ſolutio patet. Quinetiam liceret hanc expreſſionem in infinitum extendere, ſi quælibet formula indeterminata aliam novam formulam integralem indefinitam in ſe complectatur; neque difficultas ulla adeſſet, niſi in characterum ſufficienti numero ſuppeditando. Quæ cum ulterius proſequi non ſit neceſſe, unicum caſum principalem evolvere conveniet, quo

Digitized by Google

in formula $\int [Z]\,dx$, quæ valorem ipſius Π præbet, ipſa quantitas $[Z]$ denuo Π involvit. Hoc enim caſu complexio iſtiuſmodi formularum integralium actu in infinitum progreditur; namque ſi ſit $d[Z] = [L]d\Pi + [M]dx + [N]dy + [P]dp + [Q]dq +$ &c. erit hic iterum $d\Pi$, quod ante fuerat $d\pi$, & quoniam eſt $d\Pi = [Z]\,dx$, denuo eadem æquatio $d[Z] = [L]d\Pi + [M]dx + [N]dy +$ &c. recurrit, atque ita tractatio formularum integralium nuſquam abrumpetur. Caſum igitur hunc, cum quia inſignem nobis afferet uſum, tum quia concinnam admittit ſolutionem, pertractabimus.

PROPOSITIO V. PROBLEMA.

38. *Si* Π *aliter non detur niſi per æquationem differentialem* $d\Pi = [Z]\,dx$, *in qua* $[Z]$, *præter quantitates ad curvam pertinentes* x, y, p, q, r, *&c. ipſam quantitatem* Π *complectatur, ita ut ſit* $d[Z] = [L]d\Pi + [M]dx + [N]dy + [P]dp + [Q]dq +$ &c. *Sit* Z *functio quæcunque ipſius* Π *& ipſarum* x, y, p, q, *&c. ita ut ſit* $dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq +$ *&c. invenire curvam, in qua, pro data abſciſſa* $AZ = a$, *maximum minimumve ſit formula* $\int Z\,dx$. Fig. 4.

SOLUTIO.

Ponamus differentialia, quæ tam in Z quam in $[Z]$ inſunt, ſecundum gradum non excedere, ita ut particula $n\nu$ ultra abſciſſæ punctum L verſus initium nullam mutationem inferat. Solutio enim nihilominus hinc poterit maxime generalis confici. Sit igitur abſciſſa $AZ = x$, & applicata $LI = y$, patietur $\int Z dx$ ab adjecta particula $n\nu$ applicatæ $Nn = y''$ nullam mutationem, ejuſque valor differentialis erit $= 0$. Quamobrem valor differentialis formulæ $\int Z dx$, quatenus ad totam abſciſſam AZ extenditur, colligi debebit ex elementis Zdx, $Z'dx$, $Z''dx$, $Z'''dx$, &c. Singulorum autem horum elementorum valores differentiales invenientur, ſi ea differentientur, & loco differentialium dy, dy', dy'', dp, dp', dp'', & dq, dq', dq'' valores

valores §.30 indicati ſubſtituantur. Quoniam autem inſuper in hæc differentialia ingrediuntur $d\Pi$, $d\Pi'$, $d\Pi''$, &c. ponamus eorum valores ex $n\nu$ oriundos tantiſper, donec eos inveniamus, eſſe hos :

$d\Pi = n\nu.$	$d\Pi''' = n\nu.\ \delta$	$d\Pi^{VI} = n\nu.\ \eta$
$d\Pi' = n\nu.\ \beta$	$d\Pi^{IV} = n\nu.\ \epsilon$	$d\Pi^{VII} = n\nu.\ \theta$
$d\Pi'' = n\nu.\ \gamma$	$d\Pi^{V} = n\nu.\ \zeta$	. &c.

Hinc itaque erunt valores differentiales

$$d.Z dx = n\nu.\ dx\ (L\alpha + \frac{Q}{dx^2})$$
$$d.Z' dx = n\nu.\ dx\ (L'\beta + \frac{P'}{dx} - \frac{2Q'}{dx^2})$$
$$d.Z'' dx = n\nu.\ dx\ (L''\gamma + N'' - \frac{P''}{dx} + \frac{Q''}{dx^2})$$
$$d.Z''' dx = n\nu.\ dx\ L'''\delta$$
$$d.Z^{IV} dx = n\nu.\ dx\ L^{IV}\epsilon$$
$$d.Z^{V} dx = n\nu.\ dx.\ L^{V}\zeta$$
&c.

Ut nunc valores litterarum α, β, γ; δ, ϵ, &c. definiamus; notandum eſt eſſe $d\Pi$, $d\Pi'$, $d\Pi''$, &c. valores differentiales quantitatum Π, Π', Π'', &c. Eſt vero

$$\Pi = \int [Z]\, dx$$
$$\Pi' = \int [Z] dx + [Z] dx$$
$$\Pi'' = \int [Z] dx + [Z] dx + [Z'] dx$$
$$\Pi''' = \int [Z] dx + [Z] dx + [Z'] dx + [Z''] dx$$
&c.

ubi $\int [Z] dx$, per hypotheſin, a particula $n\nu$ non afficitur. Valores igitur differentiales formularum $[Z] dx$, $[Z'] dx$, $[Z''] dx$ &c. ſunt inveſtigandi, qui erunt

 $d.[Z] dx$

$$
\begin{aligned}
d.[Z]dx &= nv.dx\left([L]\alpha + \frac{[Q]}{dx^2}\right)\\
d.[Z']dx &= nv.dx\left([L']\beta + \frac{[P]}{dx} - \frac{2[Q']}{dx^2}\right)\\
d.[Z'']dx &= nv.dx\left([L'']\gamma + [N''] - \frac{[P'']}{dx} + \frac{[Q'']}{dx^2}\right)\\
d.[Z''']dx &= nv.dx\,[L''']\delta\\
d.[Z^{IV}]dx &= nv.dx.\,[L^{IV}]\epsilon\\
d.[Z^{V}]dx &= nv.dx.\,[L^{V}]\zeta
\end{aligned}
$$

&c.

Ex his igitur erit ut ſequitur

$$
\begin{aligned}
d\Pi &= \alpha\\
d\Pi' &= nv.dx\left([L]\alpha + \frac{[Q]}{dx^2}\right)\\
d\Pi'' &= nv.dx.\left([L]\alpha + [L']\beta + \frac{[P']}{dx} - \frac{[Q]+2d[Q]}{dx^2}\right)\\
d\Pi''' &= nv.dx\left([L]\alpha + [L']\beta + [L'']\gamma + [N''] - \frac{d[P']}{dx}\right.\\
&\qquad \left. + \frac{dd[Q]}{dx^2}\right)\\
d\Pi^{IV} &= nv.dx[L]\alpha + [L']\beta + [L'']\gamma + [L''']\delta + [N'']\\
&\qquad - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2}\Big)\\
d\Pi^{V} &= nv.dx\left([L]\alpha + [L']\beta + [L'']\gamma + [L''']\delta\right.\\
&\qquad \left. + [L^{IV}]\epsilon + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2}\right)
\end{aligned}
$$

&c.

His comparatis cum valoribus aſſumtis, erit

$$
\begin{aligned}
\alpha &= 0\\
\beta &= [L]\alpha\,dx + \frac{[Q]}{dx}\\
\gamma &= dx\left([L]\alpha + [L']\beta + \frac{[P']}{dx} - \frac{[Q]+2d[Q]}{dx^2}\right)
\end{aligned}
$$

$\delta =$

$$\delta = dx\,([L]\alpha + [L']\beta + [L'']\gamma + [N''] - \frac{d[P'']}{dx} + \frac{dd[Q]}{dx^2})$$

$$\varepsilon = dx\,([L]\alpha + [L']\beta + [L'']\gamma + [L''']\delta + [N''] - \frac{d[P']}{dx} + \frac{dd[Q]}{dx^2})$$

&c.

Ex hiſque æquationibus elicitur:

$$\alpha = 0$$

$$\beta = \frac{[Q]}{dx}$$

$$\gamma = [L'][Q] + [P'] - \frac{[Q] + 2d[Q]}{dx}$$

$$\delta = [L'][Q] + [L''][L'][Q]dx + [L''][P']dx - [L''][Q] - 2[L'']d[Q] + [N']dx - d[P'] + \frac{dd[Q]}{dx}$$

ſeu $\delta = [L''][L'][Q]dx + [L''][P']dx - [Q]d[L'] - 2[L'']d[Q] + [N']dx - d[P'] + \frac{dd[Q]}{dx}$

qui valor ipſius δ notetur, eritque porro

$$\varepsilon = \delta(1 + [L''']dx)$$

$$\zeta = \delta(1 + [L''']dx)(1 + [L^{IV}]dx)$$

$$\eta = \delta(1 + [L''']dx)(1 + [L^{IV}]dx)(1 + [L^{V}]dx)$$

&c.

Cognitis his valoribus, erit valor differentialis elementis $Zdx + Z'dx + Z''dx$ reſpondens

$= nv.\,dx\,(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} + L[L][Q] + L[P] - \frac{[Q]dL + 2Ld[Q]}{dx})$. Sequentium autem elementorum omnium uſque ad Z valor differentialis, ſi ponatur $V = [L^2][Q] + [L]$

$+ [L][P] - \frac{[Q]d[L] + 2[L]d[Q]}{dx} + N - \frac{d[P]}{dx} + \frac{dd[Q]}{dx}$, ſeu $\delta = V dx$, erit ſequens: $nv.\, dx(L''' dx + L^{IV} dx (1 + [L''']dx) + L^{V} dx (1 + [L''']dx)(1 + [L^{IV}]dx) + L^{VI} dx . (1 + [L''']dx)(1 + [L^{IV}]dx)(1 + [L^{V}]dx) + \&c.)V$.

Quamobrem hujus ſeriei ſumma eſt indaganda; hunc in finem, ſcribamus L loco L''', & $[L]$ loco $[L''']$, ſitque ſumma, quam quærimus, $= S$: erit $S = Ldx + L' dx (1 + [L]dx) + L'' dx (1 + [L]dx)(1 + [L']dx) + L''' dx (1 + [L]dx)(1 + [L']dx)(1 + [L'']dx) + \&c.$ Jam ipſius S ſumatur valor ſequens $S' = S + dS$ erit $S + dS = L'dx + L''dx(1 + [L']dx) + L'''dx(1 + [L']dx)(1 + [L'']dx) + \&c.$ Hincque $- dS = Ldx + L'[L]dx^2 + [L]dx . L''dx (1 + [L']dx) + [L]dx . L''' dx (1 + [L']dx)(1 + [L'']dx) + \&c.$ quæ ſeries cum ad priorem reduci queat, erit $- dS = Ldx + S[L]dx$, ſive ob $S' = S$, $dS + S[L]dx = - Ldx$; quæ integrata dat $e^{\int [L]dx} S = C - \int e^{\int [L]dx} L dx$; quæ conſtans C ita debet accipi, ut poſito $x = a$ fiat $S = 0$. Hanc ob rem erit valor illius ſeriei $S = e^{-\int [L]dx} (C - \int e^{\int [L]dx} L dx)$. Ex his igitur formulæ propoſitæ $\int Z dx$ orietur ſequens valor differentialis: $nv.\, dx (N - \frac{dP}{dx} + \frac{ddQ}{dx^2} + L[L][Q] + L[P] - \frac{[Q]dL + 2Ld[Q]}{dx} + S([L^2][Q] + [L][P] - \frac{[Q]d[L] + 2[L]d[Q]}{dx} + [N] - \frac{d[P]}{dx} + \frac{dd[Q]}{dx^2}))$, qui tranſmutatur in hanc formam commodiorem, $nv.\, dx (N - \frac{dP}{dx} + \frac{ddQ}{dx^2} + [N]S - \frac{d.[P]S}{dx} + \frac{dd.[Q]S}{dx^2})$. Hinc autem formari poteſt valor differentialis formulæ $\int Z dx$, ſi tam in Z quam in $[Z]$ differen-

tialia

tialia ad gradum quemcunque aſſurgant. Ad hoc efficiendum, ſit valor formulæ integralis $\int e^{\int [L]\,dx} L\,dx$, quem obtinet, ſi $x = a$ ponatur, $= H$, ac ſcribatur, brevitatis ergo, V loco hujus expreſſionis $e^{-\int [L]\,dx}(H - \int e^{\int [L]\,dx} L\,dx)$, eritque valor differentialis $= n\nu . dx\,(N + [N]V - \frac{d.(P+[P]V)}{dx} + \frac{dd(Q+[Q]V)}{dx^2} - \frac{d^3(R+[R]V)}{dx^3} + \&c.)$ Atque hinc pro curva quæſita orietur iſta æquatio, $0 = N + [N]V - \frac{d(P+[P]V)}{dx} + \frac{dd(Q+[Q]V)}{dx^2} - \frac{d^3(R+[R]V)}{dx^3} + \frac{d^4(S+[S]V)}{dx^4} - \&c.$ *Q. E. I.*

Coroll. I.

39. Inſervit igitur iſta propoſitio ejuſmodi Problematibus reſolvendis, in quibus maximi minimive formula $\int Z\,dx$ talem in ſe continet quantitatem Π, quæ nequidem formula integrali ex quantitatibus ad curvam pertinentibus $x, y, p, q, r,$ &c. exhiberi poteſt, ſed cujus determinatio pendet a reſolutione æquationis differentialis cujuſcunque. Habetur enim $d\Pi = [Z]\,dx$, atque $[Z]$ ipſam quantitatem Π utcunque in ſe complecti ponitur.

Coroll. II.

40. Caſus hic notari meretur, quo eſt $L = [L]$, quippe quo fit formula $\int e^{\int [L]\,dx} L\,dx$ integrabilis, integrali exiſtente $= e^{\int [L]\,dx}$. Quod ſi ergo, poſito $x = a$, abeat $e^{\int [L]\,dx}$ in H, fiet $V = He^{-\int [L]\,dx} - 1$.

COROLL. III.

41. Caſus hic potiſſimum locum habet, quando curva quæritur, in qua ſit ipſa formula $\Pi = \int [Z]\, dx$ maximum vel minimum. Tum enim fit $Z = [Z]$, & hinc $L = [L]$, $M = [M]$, $N = [N]$ &c. Hinc itaque erit valor differentialis $= nv.\, dx \left(H[N]\, e^{-\int [L]\, dx} - \frac{d.\, H[P]\, e^{-\int [L]\, dx}}{dx} + \frac{dd\, H[Q]\, e^{-\int [L]\, dx}}{dx^2} \right.$ — &c. Atque æquatio pro curva erit

$$0 = [N]\, e^{-\int [L]\, dx} - \frac{d.[P]\, e^{-\int [L]\, dx}}{dx} + \frac{dd.[Q]\, e^{-\int [L]\, dx}}{dx^2}$$

— &c.

COROLL. IV.

42. Quia ex hac æquatione quantitas H a data abſciſſa A Z $= a$ pendens per diviſionem eſt egreſſa; patet his caſibus curvam uni abſciſſæ ſatisfacientem, eandem pro omni alia abſciſſa eſſe ſatisfacturam: ita ut hæc Problemata ſimilia ſint iis, in quibus quantitas Z eſt functio determinata.

COROLL. V.

43. Si ergo quantitas $\Pi = \int [Z]\, dx$ debeat eſſe maximum vel minimum, exiſtente $d[Z] = [L]\, d\Pi + [M]\, dx + [N]\, dy + [P]\, dp + [Q]\, dq +$ &c. curva poterit exhiberi, quæ una pro quacunque abſciſſa iſta proprietate gaudeat; ejuſque natura exprimetur hac æquatione: $0 = [N]\, e^{-\int [L] dx} - \frac{d.[P]\, e^{-\int [L] dx}}{dx} + \frac{dd.[Q]\, e^{-\int [L]\, dx}}{dx^2}$ — &c. Ex qua inſuper, evolutis ſingulis terminis, quantitas exponentialis $e^{-\int [L]\, dx}$, atque adeo ipſa formula integralis $\int [L]\, dx$ excedent.

SCHO-

SCHOLION I.

44. Uſus hujus Propoſitionis eximius eſt in quæſtionibus ita comparatis, ut quantitates indefinitæ in iis contentæ per formulas integrales exhiberi nequeant, verum conſtructionem æquationum differentialium poſtulent. Atque hæc ſolutio perinde valet, ſive una hujuſmodi quantitas π inſit in formula maximi minimive $\int Z dx$ ſive plures; quod ſi enim plures inſint ejuſmodi quantitates π, plures etiam habebuntur valores litterarum L, $[L]$, $[M]$, $[N]$, $[P]$, $[Q]$, &c. atque etiam litteræ $V = e^{-\int [L] dx} (H - \int e^{\int [L] dx} L dx)$; qui omnes æqualiter, eo modo quem invenimus, in valorem differentialem formulæ $\int Z dx$ introducti præbebunt æquationem pro curva; ſimilisque omnino tractatio erit, ac ſi unica tantum adeſſet. Quoniam autem littera iſta π, cujus valor abſolutus per quantitates ad curvam pertinentes exhiberi non poteſt, in omnibus fere terminis manet; æquatio pro curva, quæ invenitur, non ſolum ex litteris x, y, p, q, r, &c. conſtabit, ſed etiam ipſam eam quantitatem π, aliaſque formulas integrales plerumque ab ea pendentes, uti $\int [L] dx$ & $\int L dx$, involvet. Quare ut æquatio pro curva pura, quæ tantum litteris x, y, p, q, &c. contineatur, prodeat, oportet cum æquatione inventa, poſtquam a formulis integralibus $\int [L] dx$ & $\int L dx$ eſt liberata, conjungi æquationem $d\pi = [Z] dx$, ejuſque ope valorem π eliminari. Quanquam autem hoc modo ad differentialia altiorum ordinum pervenitur, tamen non totidem ineſſe cenſendæ ſunt conſtantes arbitrariæ. Nam tam ipſa æquatio $d\pi = [Z] dx$, quam reliquæ anteriores æquationes, certam requirunt determinationem, unde plures conſtantes determinabuntur. Cæterum notandum eſt veritatem hujus Methodi comprobari poſſe per præcedentes, quando æquatio $d\pi = [Z] dx$ ita eſt comparata ut integrationem admittat: tum enim eædem quæſtiones per Methodos ante traditas reſolvi poterunt, indeque conſenſum obſervare licebit. Ita ſi $[Z]$ tantum ex x & π conſtet, tum certum erit π eſſe functio-

nem quamdam ipſius x determinatam, atque ſolutionem ad Caput præcedens pertinere. Idem vero hæc ſolutio patefaciet, cum enim ſit hoc caſu $[N] = 0$, $[P] = 0$, $[Q] = 0$, &c. æquatio pro curva erit $0 = N - \frac{dP}{dx} + \frac{ddQ}{dx^2} -$ &c. quæ eadem per Methodum priorem obtinetur. Uſus autem hujus ſolutionis clarius per aliquot Exempla declarabitur.

Exemplum I.

45. *Invenire curvam, in qua ſit maximus valor ipſius* Π, *exiſtente* $d\Pi = g\,dx - \alpha\Pi^n\,dx\sqrt{(1+pp)}$.

Quæſtio hæc occurrit quando quæritur curva, ſuper qua grave in medio reſiſtente ſecundum celeritatum rationem *2n plicatam* deſcendens maximam obtinet celeritatem: denotat enim Π quadratum celeritatis, & g vim gravitatis ſecundum directionem axis AZ exertam. Pertinet itaque hæc quæſtio ad caſum Coroll. 3 4, & 5 expoſitum, quo erat $Z = [Z] = g - \alpha\Pi^n\sqrt{(1+pp)}$; atque adeo curva uni abſciſſæ ſatisfaciens pro omni abſciſſa æque valebit. Cum igitur ſit $dZ = -\alpha n\Pi^{n-1}d\Pi\sqrt{(1+pp)} - \frac{\alpha\Pi^n p\,dp}{\sqrt{(1+pp)}}$, erit $[L] = -\alpha n\Pi^{n-1}\sqrt{(1+pp)}$, $[M] = 0$, $[N] = 0$, $[P] = -\frac{\alpha\Pi^n p}{\sqrt{(1+pp)}}$; $[Q] = 0$, &c. Unde pro curva quæſita iſta invenitur æquatio: $0 = -d.[P]e^{-\int[L]dx}$, ſeu $[P]e^{-\int[L]dx} = C$; hincque $-\int[L]dx = lC - l[P]$, & $[L]dx = \frac{d[P]}{[P]}$. Subſtitutis ergo loco $[L]$ & $[P]$ debitis valoribus, erit $\int\alpha n\Pi^{n-1}dx\sqrt{(1+pp)} = +lC - l-\alpha - l\Pi^n - lp + l\sqrt{(1+pp)}$; hincque $\alpha n\Pi^{n-1}dx\sqrt{(1+pp)} = -\frac{nd\Pi}{\Pi} - \frac{dp}{p} + \frac{p\,dp}{(1+pp)}$ $=$

$= -\frac{dp}{p(1+pp)} - \frac{n\,d\Pi}{\Pi}$; ſeu $0 = nd\Pi + \alpha n\Pi^n dx\sqrt{(1+pp)} + \frac{\Pi dp}{p(1+pp)}$. Quæ æquatio, ut eliminetur Π, conjungenda eſt cum hac $d\Pi + \alpha\Pi^n dx\sqrt{(1+pp)} = g\,dx$; unde ſtatim fit $0 = ngdx + \frac{\Pi dp}{p(1+pp)}$, & $\Pi = -\frac{ngpdx(1+pp)}{dp}$. Cum igitur curva fuerit inventa, hæc æquatio ſtatim præbet celeritatem corporis in quovis curvæ loco. Ponatur $dx = -\frac{tdp}{ng}$, erit $\Pi = pt(1+pp)$ & $d\Pi = pdt(1+pp) + tdp(1+3pp)$; hincque obtinebitur iſta æquatio, $pdt(1+pp) + tdp(1+3pp)$ $- \frac{\alpha p^n t^{n+1}(1+pp)^{n+\frac{1}{2}}dp}{ng} + \frac{tdp}{n} = 0$, quæ tranſmutatur in hanc $\frac{npdt(1+pp)+tdp(n+1+3npp)}{nt^{n+1}p^{n+1}(1+pp)^{n+\frac{1}{2}}} = \frac{\alpha dp}{ngp^2}$; cujus integralis eſt $\frac{1}{nt^n p^{n+1}(1+pp)^{n-\frac{1}{2}}} = \frac{\alpha}{ngp} + \frac{\beta}{ng}$, ſeu $g = (\alpha+\beta p)t^n p^n(1+pp)^{n-\frac{1}{2}}$; hincque $t = \frac{\sqrt[n]{g}}{p(1+pp)^{1-1:2n}\sqrt[n]{(\alpha+\beta p)}}$. Erit igitur $dx =$ $\frac{-dp}{np(1+pp)^{1-1:2n}\sqrt[n]{g^{n-1}(\alpha+\beta p)}}$, & $dy =$ $\frac{-dp}{n(1+pp)^{1-1:2n}\sqrt[n]{g^{n-1}(\alpha+\beta p)}}$; hincque $\Pi =$ $\sqrt[n]{\frac{g\sqrt{(1+pp)}}{\alpha+\beta p}}$. Erit ergo $x = -\frac{1}{ng}\int\frac{dp}{p(1+pp)}\sqrt[n]{\frac{g\sqrt{(1+pp)}}{\alpha+\beta p}}$, atque $y = -\frac{1}{ng}\int\frac{dp}{1+pp}\sqrt[n]{\frac{g\sqrt{(1+pp)}}{\alpha+\beta p}}$.

Hinc apparet quantitatem Π ſuper curva nuſquam eſſe poſſe $= 0$; hanc ob rem, in curvæ initio Π jam habebit certum quem-

Digitized by Google

dam valorem. Ut autem indoles curvæ magis percipiatur, ex æquatione $\Pi = -\frac{ngp\,dx(1+pp)}{dp}$ patet valorem ipſius dp ubique negativum eſſe oportere, ex quo curva verſus axem erit concava. Quia igitur valores ipſius p recedendo a curvæ initio decreſcunt, in ipſo curvæ initio p maximum habebit valorem. Hinc ponamus initium curvæ ibi, ubi eſt $p = \infty$. Sit ergo AP axis curvæ verticalis, in cujus directione vis gravitatis g corpus deorſum trahat, atque in initio curvæ A ſit tangens horizontalis Aa: ibique corpus motum ſuper curva incipiat, celeritate, cujus quadratum ſit $= b$. Erit igitur, poſito $p = \infty$, $b = \sqrt[n]{\frac{g}{\mathfrak{C}}}$, atque $\mathfrak{C}b^n = g$, ſeu $\mathfrak{C} = \frac{g}{b^n}$. Porro ad uniformitatem conſervandam ſit $\alpha = \frac{1}{k^n}$. Quod ſi jam curva quæſita ſit AM, & ponatur $AP = x$, $PM = y$, & $dy = pdx$; erit in M celeritatis quadratum $\Pi = bk\sqrt[n]{\frac{g\sqrt{(1+pp)}}{b^n + gk^n p}}$; atque ubi tangens curvæ fiet verticalis, ibi erit celeritatis quadratum $= k\sqrt[n]{g}$. Curvæ autem conſtructio ita conficietur, ut ſit

Fig. 6.

$$x = -\frac{bk}{ng}\int\frac{dp}{p(1+pp)}\sqrt[n]{\frac{g\sqrt{(1+pp)}}{b^n + gk^n p}} \;\&$$

$$y = -\frac{bk}{ng}\int\frac{dp}{1+pp}\sqrt[n]{\frac{g\sqrt{(1+pp)}}{b^n + gk^n p}}.$$

Deinde commemorari meretur ſingularis proprietas, ſeu relatio inter corporis deſcendentis vim centrifugam, quæ eſt $\frac{2\Pi}{\text{rad. oſculi}}$, & vim normalem quæ eſt $\frac{gp}{\sqrt{(1+pp)}}$. Quod ſi enim vis centrifuga $\frac{2\Pi}{\text{rad. oſc.}} = \frac{-2\Pi\,dp}{dx(1+pp)^{3:2}}$ ponatur $= F$, & vis normalis $\frac{gp}{\sqrt{(1+pp)}} = G$; erit, ex æquatione $\Pi = -\frac{ngp\,dx(1+pp)}{dp}$, ſeu $\frac{-2\Pi\,dp}{dx(1+pp)^{3:2}} = \frac{2ngp}{\sqrt{(1+pp)}}$, hæc relatio

tio inter vim centrifugam F & vim normalem G, ut ſit $F = 2nG$: nempe vis normalis ſe habebit ad vim centrifugam ut 1 ad $2n$. Corpus in A data celeritate motum inchoans deſcendendo ſuper curva AM, in quovis loco M abſciſſæ AP reſpondente majorem habebit celeritatem, quam ſi ſuper alia quacunque curva eadem celeritate initiali deſcendiſſet. Evolvamus autem binos caſus principales;

Sitque 1°. reſiſtentia quadratis celeritatum proportionalis; erit $n = 1$, & $F = 2G$. Pro curva autem habebitur:

$$x = -bk\int \frac{dp}{p(b+gkp)\sqrt{(1+pp)}}$$

$$\& \; y = -bk\int \frac{dp}{(b+gkp)\sqrt{(1+pp)}};$$

itemque arcus curvæ AM $= -bk\int \frac{dp}{p(b+gkp)} = C + kl\frac{b+gkp}{p}$. Ponatur arcus AM $= s$, qui cum evaneſcere debeat poſito $p = \infty$, erit $s = kl\frac{b+gkp}{gkp}$, hincque $e^{s:k}gkp = b+gkp$, & $p = \frac{b}{gk(e^{s:k}-1)} = \frac{dy}{dx}$. Unde oritur $bdx + gkdy = gke^{s:k}dy$. Erit autem porro ex æquatione $y = -bk\int \frac{dp}{(b+gkp)\sqrt{(1+pp)}}$ integrata $y = \frac{bk}{\sqrt{(bb+ggkk)}}\, l\frac{(b+gkp)(b+\sqrt{(bb+ggkk)})}{gk(bp-gk+\sqrt{(bb+ggkk)(1+pp)})}$.

2°. Sit reſiſtentia ipſis celeritatibus proportionalis, fiet $n = \frac{1}{2}$ & $F = G$, hoc eſt vis centrifuga vi normali erit æqualis. Quæ binæ vires cum ſint contrariæ, quæſito ſatisfaciet ea curva, quæ a corpore ſuper ea deſcendente omnino non premitur. Erit autem

$$x = -2gbk\int \frac{dp}{p(\sqrt{b}+gp\sqrt{k})^2}$$

$$\& \; y = -2gbk\int \frac{dp}{(\sqrt{b}+gp\sqrt{k})^2} = \frac{2b\sqrt{k}}{\sqrt{b}+gp\sqrt{k}};$$

hincque $ydx\sqrt{b} + gydy\sqrt{k} = 2bdx\sqrt{k}$, & $dx = \frac{gydy\sqrt{k}}{2b\sqrt{k}-y\sqrt{b}}$; hincque integrando $x = -gy\sqrt{\frac{k}{b}} + 2gkl\frac{2b\sqrt{k}}{2b\sqrt{k}-y\sqrt{b}}$. Hæc ergo

go curva non ſolum per Logarithmicam conſtrui poteſt, verum eſt portio ipſius Logarithmicæ obliquangulæ. Erit ſcilicet ipſa curva projectoria, quam corpus in hac reſiſtentiæ hypotheſi projectum libere deſcribit. Hæc convenientia ex eo patet, quod curva a corpore moto nullam ſuſtinet preſſionem, quæ eſt proprietas curvarum libere deſcriptarum.

EXEMPLUM II.

46. *Invenire curvam in qua, pro data abſciſſa* $x = a$, *minimum ſit iſta formula* $\int \frac{dx\sqrt{(1+pp)}}{\sqrt{\Pi}}$, *exiſtente* $d\Pi = g\,dx - \alpha \Pi^n dx \sqrt{(1+pp)}$.

Quæſtio hæc congruit cum illa, in qua requiritur curva, ſuper qua corpus deſcendens, in medio reſiſtente cujus reſiſtentia eſt ut poteſtas exponentis $2n$ celeritatis, citiſſime arcum abſciſſæ a reſpondentem abſolvit. Denotat enim hîc g vim gravitatis ſecundum directionem axis ſollicitantem, $\sqrt{\Pi}$ celeritatem corporis in quocunque loco, & $\alpha \Pi^n$ reſiſtentiam medii ipſam. Erit itaque $Z = \frac{\sqrt{(1+pp)}}{\sqrt{\Pi}}$, & hinc $dZ = \ldots$ $\frac{-d\Pi\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}} + \frac{p\,dp}{\sqrt{\Pi(1+pp)}}$, unde erit $L = \frac{-\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}}$; $M = 0$, $N = 0$, $P = \frac{p}{\sqrt{\Pi(1+pp)}}$ Porro erit $[Z] = g - \alpha\Pi^n \sqrt{(1+pp)}$, & $d[Z] = -\alpha n \Pi^{n-1} d\Pi \sqrt{(1+pp)} - \frac{\alpha \Pi^n p\,dp}{\sqrt{(1+pp)}}$; unde erit $[L] = -\alpha n \Pi^{n-1} \sqrt{(1+pp)}$; $[M] = 0$, $[N] = 0$, & $[P] = \frac{-\alpha \Pi^n p}{\sqrt{(1+pp)}}$. Habebitur ergo $V = e^{\alpha n \int \Pi^{n-1} dx \sqrt{(1+pp)}} \ldots\ldots$ $\times(\int e^{-\alpha n \int \Pi^{n-1} dx \sqrt{(1+pp)}} \frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}} - H)$;

deno-

denotante H eum valorem formulæ . . . . $\int e^{-\alpha n\int \Pi^{n-1} dx\sqrt{(1+pp)}} \frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}}$ quem obtinet ſi fit $x = a$. Namque V evaneſcere debet poſito $x = a$, eſtque $dV = \alpha n V\Pi^{n-1} dx\sqrt{(1+pp)} + \frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}}$. Ex his pro curva quæſita obtinebitur iſta æquatio: $d.(P + [P]V) = 0$, & $P + [P]V = C$, ſeu $V = \frac{C - P}{[P]}$. Subſtitutis ergo valoribus debitis, erit $e^{\alpha n\int \Pi^{n-1} dx\sqrt{(1+pp)}}$ . . . . . . $\times(\int e^{-\alpha n\int \Pi^{n-1} dx\sqrt{(1+pp)}} \frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}} - H) = \frac{p - C\sqrt{\Pi(1+pp)}}{\alpha \Pi^{n} p\sqrt{\Pi}}$. Quare conſtantem C ita determinari oportet, ut poſito $x = a$, fiat $C = \frac{p}{\sqrt{\Pi(1+pp)}}$. Cum autem ſit $V = \frac{1}{\alpha\Pi^{n}\sqrt{\Pi}} - \frac{C\sqrt{(1+pp)}}{\alpha\Pi^{n} p}$, erit $dV = \frac{-(n+\frac{1}{2})d\Pi}{\alpha\Pi^{n+1}\sqrt{\Pi}} + \frac{nC\,d\Pi\sqrt{(1+pp)}}{\alpha\Pi^{n+1}p} + \frac{C\,dp}{\alpha\Pi^{n}p^{2}\sqrt{(1+pp)}} = \frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}} + \frac{n\,dx\sqrt{(1+pp)}}{\Pi\sqrt{\Pi}} - \frac{nC(1+pp)dx}{p\Pi}$; in ſubſidium vocata æquatione $dV = \alpha n V\Pi^{n-1} dx\sqrt{(1+pp)} + \frac{dx\sqrt{(1+pp)}}{2\Pi\sqrt{\Pi}}$. Cum autem ſit $d\Pi = g\,dx - \alpha\Pi^{n} dx \,.\times\sqrt{(1+pp)}$ erit $-\frac{(n+\frac{1}{2})g\,dx}{\alpha\Pi^{n+1}\sqrt{\Pi}} + \frac{nCg\,dx(1+pp)}{\alpha\Pi^{n+1}p}$ . . . $+ \frac{C\,dp}{\alpha\Pi^{n}p^{2}\sqrt{(1+pp)}} = 0$, ſeu $\frac{C\,dp}{p^{2}\sqrt{(1+pp)}} = \frac{(n+\frac{1}{2})g\,dx}{\Pi\sqrt{\Pi}} - \frac{nCg\,dx\sqrt{(1+pp)}}{\Pi p}$. Quod ſi jam hæc æquatio cum illa $d\Pi = g\,dx - \alpha\Pi^{n} dx\sqrt{(1+pp)}$ conjungatur, poterit eliminari

minari quantitas Π, hocque pacto inveniri æquatio pro curva quæſita. Hoc autem modo calculus fieret maxime tædioſus, ac minime tractabilis. Adminiculum vero ſummum afferet ultima æquatio in hanc formam tranſmutata: $\frac{C\,dp}{gp^2} = \frac{(n+\frac{1}{2})dx\sqrt{(1+pp)}}{\Pi\sqrt{\Pi}}$ $- \frac{nC\,dx(1+pp)}{\Pi\,p}$, cui expreſſioni ante æqualis eſſe inventus eſt valor ipſius dV; erit ergo $dV = \frac{C\,dp}{gpp}$ & $V = D -$ $\frac{C}{gp} = \frac{1}{\alpha\,\Pi^n\sqrt{\Pi}} - \frac{C\sqrt{(1+pp)}}{\alpha\,\Pi^n\,p}$. Jam igitur habemus duas æquationes has $\frac{C\,dp}{gp^2} = \frac{(n+\frac{1}{2})\,dx\sqrt{(1+pp)}}{\Pi\sqrt{\Pi}}$ . . . . . . $- \frac{nC\,dx(1+pp)}{\Pi p}$, & $\alpha D - \frac{\alpha C}{gp} = \frac{1}{\Pi^n\sqrt{\Pi}} - \frac{C\sqrt{(1+pp)}}{\Pi^n\,p}$. Ex quibus ſi climinetur Π, habebitur æquatio inter p & x ejuſmodi, ut nuſquam x ſed ubique tantum dx occurrat, ex quo illa æquatio poterit conſtrui atque adeo ipſa curva. Vel facilius ex poſteriori æquatione determinetur p per Π, hicque valor in æquatione fundamentali $dx = \frac{d\Pi}{g - \alpha\,\Pi^n\sqrt{(1+pp)}}$ ſubſtitutus, dabit valorem ipſius x per Π, erit ſcilicet $x =$ $\int \frac{d\Pi}{g - \alpha\,\Pi^n\sqrt{(1+pp)}}$ atque $y = \int \frac{p\,d\Pi}{g - \alpha\,\Pi^n\sqrt{(1+pp)}}$. Conſtans autem D ita debet accipi, ut poſito $x = a$, quo caſu fit $C = \frac{p}{\sqrt{\Pi(1+pp)}}$, fiat $D = \frac{1}{g\sqrt{\Pi(+pp)}}$, ſeu tum eſſe debet $\frac{C}{D} = gp$.

SCHOLION II.

47. In his igitur duobus Capitibus, Methodum expoſuimus inveniendi lineam curvam, in qua, pro datæ magnitudinis abſciſſa $= a$, maximum minimumve ſit formula $\int Z\,dx$, exiſtente Z func-

functione ipſarum x, y, p, q, r, &c. ſive determinata ſive indeterminata. Functio autem determinata nobis eſt, quæ, ſi alicubi dentur valores litterarum x, y, p, q, r &c. ipſa aſſignari poteſt, ſive algebraïce ſive tranſcendenter. Functio autem indeterminata eſt, quæ per datos iſtarum litterarum valores, quos uno in loco obtinent, aſſignari nequit, ſed omnes valores præcedentes ſimul involvit, quemadmodum hoc evenit, ſi ſigna integralia occurrant. In Capite igitur ſecundo Methodum tradidimus omnia Problemata reſolvendi, in quibus Z eſt functio determinata; in tertio vero hoc Capite perſecuti ſumus eas formulas, in quibus Z, vel ipſa eſt functio indefinita, vel talium unam pluresve involvit; ſimulque Methodum exhibuimus pro iis caſibus, quibus functio illa indefinita nequidem per formulas integrales repræſentari poteſt, verum reſolutionem æquationis differentialis requirit. Nunc igitur eos caſus evolvamus, in quibus expreſſio, quæ maximum minimumve eſſe debet, non ſimplex eſt formula integralis, uti hactenus poſuimus, ſed ex pluribus ejuſmodi formulis utcunque compoſita: atque ſimul Methodum aperiemus plura alia Problemata, quæ non ad coordinatas orthogonales ſpectant, expedite reſolvendi.

CAPUT IV.

De Uſu Methodi hactenus traditæ in reſolutione varii generis quæſtionum.

PROPOSITIO I. PROBLEMA.

1. *Invenire æquationem inter binas variabiles* x *&* y, *ita ut, pro dato ipſius* x *valore, puta poſito* $x = a$, *formula* $\int Z\,dx$ *obtineat maximum minimumve valorem, exiſtente* Z *functione ipſarum* x, y, p, q, r, *&c. ſive determinata ſive indeterminata.*

SOLUTIO.

Ex quacunque conſideratione variabiles x & y ſint natæ, ex

ſemper tanquam coordinatæ orthogonales cujuſpiam curvæ conſiderari poſſunt: atque hanc ob rem quæſtio propoſita huc revocatur, ut determinetur curva abſciſſam habens $= x$ & applicatam $= y$, in qua valor $\int Z dx$, ſi ad abſciſſam datæ magnitudinis, puta $x = a$, applicetur, fiat omnium maximus vel minimus. Quod ſi autem Problema hoc modo proponatur, tum ejus ſolutio in præcedentibus Capitibus ſatis ſuperque eſt tradita. Quamobrem formulæ propoſitæ $\int Z dx$, ſecundum Methodos ante expoſitas, capi oportet valorem differentialem, qui datæ abſciſſæ $x = a$ conveniat, iſque nihilo æqualis poſitus dabit æquationem inter x & y deſideratam, quæ pro data abſciſſa $x = a$, producet formulæ $\int Z dx$ maximum minimumve valorem. Q. E. I.

COROLL. I.

2. Methodus ergo ante tradita multo latius patet; quam ad æquationes inter coordinatas curvarum inveniendas, ut quæpiam expreſſio $\int Z dx$ fiat maximum minimumve. Extenditur ſcilicet ad binas quaſcunque variabiles, ſive eæ ad curvam aliquam pertineant quomodocunque, ſive in ſola analytica abſtractione verſentur.

COROLL. II.

3. Inter binas autem variabiles propoſitas diſcrimen ingens intercedit, eo quod propoſita formula $\int Z dx$ pro determinato quodam alterius variabilis valore maximum minimumve obtinere debeat. Iſthanc ergo variabilem conſtanter litera x, alteram vero littera y denotari convenit.

COROLL. III.

4. Litteris igitur x & y debito modo binis quantitatibus variabilibus impoſitis, erit $p = \frac{dy}{dx}$, $q = \frac{dp}{dx}$, $r = \frac{dq}{dx}$;

$s =$

$s = \frac{dr}{dx}$ &c. His ſcilicet litteris differentialia cujuſcunque gradus, quæ forte in maximi minimive formula inſint, tolli poterunt, ita ut Z futura ſit functio litterarum x, y, p, q, r, &c.

Coroll. IV.

5. Cum ergo maximi minimive formula ad talem formam $\int Zdx$ fuerit reducta, in qua Z ſit functio ipſarum x, y, p, q, r, &c. ſive definita ſive indefinita, tum ex ſuperioribus præceptis formulæ $\int Zdx$ valor differentialis, reſpondens toti abſciſſæ propoſitæ $x = a$, debet inveſtigari, qui nihilo æqualis poſitus præbebit æquationem inter x & y quæſitam.

Coroll. V.

6. Si Z eſt functio definita ipſarum x, y, p, q, r, &c. tum valor differentialis formulæ $\int Zdx$ non pendet a præſcripto abſciſſæ valore $x = a$; & hanc ob rem æquatio inter x & y inventa pro qualibet abſciſſa præbebit maximum vel minimum formulæ $\int Zdx$.

Scholion I.

7. Quia in hoc negotio valores differentiales, quos ante pro omni genere formularum ſparſim eruimus, in promtu eſſe oportet, eos hic conjunctim in conſpectum producemus, ut ſit unde, quovis caſu oblato, valores differentiales, quibus opus fuerit, conquiri ac depromi queant. Exhibebimus igitur formulæ $\int Zdx$ pro varia functionis Z indole valorem differentialem, qui perpetuo determinatæ variabilis x quantitati, puta $x = a$, reſpondeat.

Digitized by Google

I.

Maximi minimive formula

$$\int Z\,dx.$$

$$dZ = M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + \&c.$$

Valor differentialis erit

$$nv.\,dx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3 R}{dx^3} + \frac{d^4 S}{dx^4} - \&c.\right)$$

qui valor differentialis pro omni variabilis x magnitudine æque valet.

II.

Maximi minimive formula

$$\int Z\,dx.$$

$$dZ = L\,d\Pi + M\,dx + N\,dy + P\,dp + Q\,dq + \&c.$$

$$\& \; \Pi = \int [Z]\,dx$$

exiſtente

$$d[Z] = [M]\,dx + [N]\,dy + [P]\,dp + [Q]\,dq + [R]\,dr + \&c.$$

Jam poſito poſt integrationem $x = a$, ſit $\int L\,dx = H$, ponaturque $H - \int L\,dx = V$,

Valor differentialis erit

$$nv.\,dx\left(N + [N]V - \frac{d.(P + [P]V)}{dx} + \frac{dd.(Q + [Q]V)}{dx^2} - \frac{d^3.(R + [R]V)}{dx^3} + \frac{d^4.(S + [S]V)}{dx^4} = \&c.\right)$$

III.

Maximi minimive formula

$$\int Z\,dx.$$

$$dZ = L\,d\Pi + M\,dx + N\,dy + P\,dp + Q\,dq + \&c.$$

$$\& \; \Pi = \int [Z]\,dx$$

$$d[Z] = [L]\,d\pi + [M]\,dx + [N]\,dy + [P]\,dp + [Q]\,dq + \&c.$$

$$\& \; \pi = \int [z]\,dx$$

$$d[z] = [m]\,dx + [n]\,dy + [p]\,dp + [q]\,dq + [r]\,dr + \&c.$$

Sit iterum, poſito poſt integrationem $x = a$ ut ante, $\int L\,dx = H$

$= H$, ac ponatur $H - \int L\,dx = V$. Jam integretur $\int [L]\,V\,dx$, ſitque integrale, eo caſu quo $x = a$ ponitur, $= G$, ac ponatur $G - \int [L]\,V\,dx = [V] = G - \int [L]\,dx\,(H - \int L\,dx)$; His poſitis, Valor differentialis erit

$$n\nu . dx\,(N + [N]V + [n][V] - \frac{d.(P + [P]V + [p][V])}{dx} + \frac{dd.(Q + [Q]V + [q][V])}{dx^2} - \frac{d^3.(R + [R]V + [r][V])}{dx^3} + \frac{d^4.(S + [S]V + [s][V])}{dx^4} - \&c.\,)$$

unde ſimul lex progreſſionis patet, ſi adhuc plura integralia involvantur.

IV.

Maximi minimive formula

$$\int Z\,dx.$$

$$dZ = L\,d\Pi + M\,dx + N\,dy + P\,dp + Q\,dq + \&c.$$

$$\& \ \Pi = \int Z\,dx$$

Abeat, poſito $x = a$, hæc expreſſio $e^{\int L\,dx}$ in H, denotante e numerum cujus logarithmus eſt $= 1$, ſitque $H\,e^{-\int L dx} = V$: Valor differentialis erit

$$n\nu . dx\,(NV - \frac{d.PV}{dx} + \frac{dd.QV}{dx^2} - \frac{d^3\,RV}{dx^3} + \frac{d^4\,SV}{dx^4} - \&c.\,)$$

V.

Maximi minimive formula

$$\int Z\,dx.$$

$$dZ = L\,d\Pi + M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + \&c.$$

$$\& \ \Pi = \int [Z]\,dx$$

$$d[Z] = [L]d\Pi + [M]dx + [N]dy + [P]dp + [Q]dq + [R]dr + \&c.$$

Sit, ſi ponatur $x = a$, poſt integrationem

$$\int e^{\int [L]\,dx}\,L\,dx = H$$

atque ponatur

Digitized by Google

$$e^{-\int [L]dx}\left(H - \int e^{\int [L]dx} L\,dx\right) = V$$

Valor differentialis erit

$$nv.\,dx\left(N + [N]V - \frac{d.(P+[P]V)}{dx} + \frac{dd(Q+[Q]V)}{dx^2} - \frac{d^3.(R+[R]V)}{dx^3} + \frac{d^4.(S+[S]V)}{dx^4} - \&c.\right).$$

In his igitur quinque caſibus continentur omnes regulæ, quas in Capitibus præcedentibus invenimus. Iique tam late patent, ut omnes caſus qui quidem occurrere queant, in iis vel actu contineantur, vel ſaltem per eos non difficulter reſolvi poſſint. Iis igitur hic in compendium redactis, eorum uſum monſtrabimus, in reſolvendis quæſtionibus, in quibus x & y non denotant coordinatas orthogonales.

EXEMPLUM I.

Fig. 7. 8. *Ex dato centro* C *ductis radiis* CA, CM, *invenire lineam* AM, *quæ inter omnes alias lineas intra angulum* ACM *contentas ſit breviſſima.*

Patet quidem hanc lineam quæſitam eſſe rectam: interim tamen hanc quæſtionem ſecundum præcepta data reſolvi conveniet, ut conſenſus Methodi cum veritate luculentius perſpiciatur. Cum igitur longitudo lineæ AM pro dato angulo ACM debeat eſſe minima; ponamus angulum hunc ACM eſſe $= x$; ſeu centro C, radio CB $= 1$, deſcribamus circulum, ſitque arcus BS $= x$. Tum ſit radius CM altera variabilis $= y$, æquatione enim inter has variabiles x & y inventa, innoteſcet natura lineæ quæſitæ AM. Jam autem ducto radio proximo Cm erit Ss $= dx$, & mn $= dy$, ſumto Cn $=$ CM: ob triangula vero ſimilia CSs & CMn erit $1 : dx =$ CM$[y]$: Mn$[y\,dx]$. Ex his itaque erit Mm $= \sqrt{(dy^2 + y^2 dx^2)}$; & quia perpetuo ponimus $dy = p\,dx$, erit Mm $= dx\sqrt{(yy+pp)}$; unde lineæ AM longitudo erit $= \int dx\sqrt{(yy+pp)}$, quæ debet eſſe minima pro dato ipſius x valore, puta $x = a$. At quia hæc formula

mula ad caſum primum pertinet, linea ſatisfaciens erit pro quovis valore ipſius x minima. Cum igitur ſit $Z = \sqrt{(yy+pp)}$, erit $dZ = \frac{ydy}{\sqrt{(yy+pp)}} + \frac{pdp}{\sqrt{(yy+pp)}}$, & in caſu primo fiet $M = 0$, $N = \frac{y}{\sqrt{(yy+pp)}}$, $P = \frac{p}{\sqrt{(yy+pp)}}$, $Q = 0$, $R = 0$, &c. ideoque $dZ = Ndy + Pdp$. Habebitur ergo iſte valor differentialis $nv.dx(N - \frac{dP}{dx})$, indeque pro ſolutione iſta æquatio, $0 = N - \frac{dP}{dx}$: quæ, multiplicata per $pdx = dy$, dat $Ndy = pdP$; quo in æquatione $dZ = Ndy + Pdp$ ſubſtituto, prodibit $dZ = Pdp + pdP$, & integrando $Z + C = Pp$, ſeu $C + \sqrt{(yy+pp)} = \frac{pp}{\sqrt{(yy+pp)}}$. Quocirca erit $\frac{yy}{\sqrt{(yy+pp)}} = Conſt. = b$. At eſt Mm $[dx\sqrt{(yy+pp)}]$: Mn $[ydx]$ = MC $[y]$: $\frac{yy}{\sqrt{(yy+pp)}}$; quæ quarta proportionalis præbet perpendiculum CP, quod ex C in tangentem lineæ quæſitæ MP demittitur. Cum igitur hoc perpendiculum CP ſit conſtans, intelligitur lineam quæſitam eſſe rectam: & quia, in æquatione inventa prima $Ndx = dP$, duæ inſunt potentia conſtantes arbitrariæ, conditio hæc quæſtioni eſt addenda, ut linea quæſita per data duo puncta tranſeat; tum igitur linea recta per illa duo puncta ducta quæſito ſatisfaciet.

EXEMPLUM II.

9. *Super axe* AP *conſtruere lineam* BM, *ita comparatam, ut, abſciſſa area* ABMP *datæ magnitudinis, arcus curvæ* BM *illi areæ reſpondens ſit omnium minimus.* Fig. 8.

Quia pro data area ABMP minima longitudo arcus BM requiritur, area ABMP nobis deſignanda erit variabili x: altera variabili y autem indicemus applicatam curvæ PM. Jam ſit abſciſſa AP $= t$, erit $x = \int y\,dt$, ideoque $dt = \frac{dx}{y}$: atque

arcus

arcus BM longitudo erit $= \int \surd (dy^2 + \frac{dx^2}{yy})$. Poſito ergo $dy = p\,dx$, minimum eſſe debet hæc formula $\int dx \surd (\frac{1}{yy} + pp) = \int \frac{dx \surd (1 + yypp)}{y}$. Erit itaque $Z = \frac{\surd (1 + yypp)}{y}$, & $dZ = -\frac{dy}{yy \surd (1 + y^2 p^2)} + \frac{yy\,p\,dp}{y \surd (1 + y^2 p^2)}$: unde $M = 0$ $N = \frac{-1}{yy \surd (1 + y^2 p^2)}$; $P = \frac{yp}{\surd (1 + y^2 p^2)}$, $Q = 0$ &c. Pertinet ergo hæc quæſtio ad caſum primum, ac ſolutio præbebit lineam curvam, quæ pro area quacunque A P M B abſciſſæ erit breviſſima. Pervenietur autem, uti in præcedente Exemplo, ad æquationem hanc $Z = C + Pp$, atque curva quæſita per data duo puncta deſcribi poterit. Erit itaque $\frac{\surd (1 + yypp)}{y} = C + \frac{ypp}{\surd (1 + yypp)}$, ſeu $1 = Cy \surd (1 + yypp)$: vel $b = y \surd (1 + yypp)$; hinc fit $bb = yy + y^4 pp$, & $p = \frac{\surd (bb - yy)}{yy} = \frac{dy}{dx} = \frac{dy}{ydt}$, ob $dx = ydt$. Erit igitur $dt = \frac{y\,dy}{\surd (bb - yy)}$, & $t = c \pm \surd (bb - yy)$. Quare linea quæſita erit Circulus, centro alicubi in axe AP, puta in C; aſſumto: iſque inter omnes alias curvas per eadem duo quæcunque puncta ductas, pro data reſecta area ABMP, habebit arcum BM breviſſimum.

Exemplum III.

Fig. 9. 10. *Eductis ex puncto fixo* C *radiis* CA, CM; *intra eos deſcribere curvam* AM, *quæ pro dato ſpatio* ACM *habeat arcum* AM *breviſſimum.*

Quia arcus AM minimus eſſe debet, ſi ſpatium ACM datæ magnitudinis abſcindatur; ponatur area hæc $ACM = x$, atque radius CM deſignetur altera variabili y. Jam ponatur arcus

arcus BS, radio CB $=1$ deſcriptus, $=t$; erit, ut ante vidimus, Mn $=y\,dt$, & area MCm $=\frac{1}{2}yy\,dt=dx$, unde fit $dt=\frac{2dx}{yy}$. Quia porro eſt Mm $=\sqrt{(dy^2+y^2dt^2)}=\sqrt{(dy^2+\frac{4dx^2}{yy})}$; ſit $dy=p\,dx$, minimumque eſſe debet $\int\frac{dx}{y}\sqrt{(4+ppyy)}$. Cum igitur ſit $Z=\frac{\sqrt{(4+y^2p^2)}}{y}$, erit $M=0$, $N=-\frac{4}{yy\sqrt{(4+y^2p^2)}}$, & $P=\frac{yp}{\sqrt{(4+y^2p^2)}}$, $Q=0$, &c. Hinc reſultat iſta æquatio $Z=C+Pp$; propterea quod ſit $M=0$: ideoque $\frac{\sqrt{(4+y^2p^2)}}{y}=C+\frac{ypp}{\sqrt{(4+y^2p^2)}}$, ſeu $4=Cy\sqrt{(4+yypp)}$ vel $2b=y\sqrt{(4+yypp)}$; hincque $p=\frac{2\sqrt{(bb-yy)}}{yy}=\frac{dy}{dx}=\frac{2\,dy}{yy\,dt}$; ac $dt=\frac{dy}{\sqrt{(bb-yy)}}$: itemque integrando $t=$ A ſin. $\frac{y}{b}+$ A ſin. $\frac{c}{b}=$ A ſin. $\frac{y\sqrt{(bb-cc)}+c\sqrt{(bb-yy)}}{bb}$. In AC ex S demittatur perpendiculum QS $=$ ſin. At, erit QS $=\frac{y\sqrt{(bb-cc)}+c\sqrt{(bb-yy)}}{bb}$. At ex æquatione $t+$ *Conſt.* $=$ A ſin. $\frac{y}{b}$ colligitur curva quæſita eſſe Circulus AME per punctum fixum C tranſiens. Deſcribatur enim ſuper diametro quacunque CE in C terminata Circulus CAME, arcus AM interceptus inter radios ACM pro data area ACM erit minimus. Scilicet ſi alia quæcunque curva per duo quæcunque puncta in hoc Circulo ſita deſcribatur, biniſque radiis ex C ductis area æqualis areæ ACM abſcindatur, arcus illius curvæ reſpondens perpetuo major erit quam arcus AM. Quod ut appareat, ducatur ex C ad CE normalis CD, in eamque ex S perpendiculum SQ demittatur: erit triangulum SCQ ſimile triangulo CEM, hincque CE : CM [y] $=$ CS [1]: SQ ſeu SQ $=\frac{y}{\text{CE}}=$ ſin. A. DBS, vel DBS *Fig. 9.*

 $=$

$= A$ ſin. $\frac{y}{CE}$. Poſita ergo diametro $CE = b$, & quia eſt $DBS = BS + BD = t + Conſt.$ erit $t + Conſt. = A$ ſin. $\frac{y}{b}$: quæ eſt ipſa illa proprietas, qua curvam quæſitam præditam eſſe oportere invenimus.

Exemplum IV.

Fig. 10. 11. *In ſuperficie quacunque, ſive convexa ſive concava, ducere lineam, quæ ſit intra ſuos terminos omnium breviſſima.*

Sumatur planum quodcunque ad quod ſuperficies referatur, APQ, in eoque capiatur recta AP pro axe. Jam ex lineæ quæſitæ ſingulis punctis concipiantur perpendicula in hoc planum demitti, quibus deſcribatur linea AQ, quæ erit projectio lineæ breviſſimæ in hoc planum; qua cognita, ſimul ipſa linea breviſſima in ſuperficie propoſita innoteſcet. Vocetur $AP = x$; $PQ = y$; atque cum natura ſuperficiei detur, ex datis $AP = x$ & $PM = y$ definiri poterit longitudo perpendicularis QM in planum APQ, donec ſuperficiem in M ſecet. Quod ſi ergo ponatur $QM = z$, longitudo hujus lineæ z dabitur per x & y, ita ut z ſit functio definita ipſarum x & y. Cum igitur ſit z functio ipſarum x & y, quæ ex æquatione locali ad ſuperficiem datur, ponamus eſſe $dz = Tdx + Vdy$; eruntque T & V ejuſmodi functiones ipſarum x & y, ut $Tdx + Vdy$ ſit formula differentialis definita: poſito nempe $dT = Edx + Fdy$, erit $dV = Fdx + Gdy$, exiſtente littera F utrique differentiali communi. Nunc elementum lineæ in ſuperficie ductæ eſt $= \surd(dx^2 + dy^2 + dz^2) = \surd(dx^2 + dy^2 + (Tdx + Vdy)^2)$. Poſito ergo $dy = pdx$, minimum eſſe debet hæc formula $\int dx \surd(1 + p^2 + T^2 + 2TVp + V^2p^2)$; ita ut ſit $Z = \surd(1 + p^2 + T^2 + 2TVp + V^2p^2)$, unde ſit

+

$$dZ = \left\{ \frac{\begin{array}{l} + TEdx + TFdy + pdp \\ + VEpdx + VFpdy + TVdp \\ + TFpdx + TGpdy + V^2pdp \\ + VFp^2dx + VGppdy \end{array}}{\sqrt{(1+pp+T^2+2TVp+V^2p^2)}} \right.$$

Quæ formula cum ad caſum primum pertineat, proveniet iſta æquatio inter x & y;

$$\frac{TFdx + VFpdx + TGpdx + VGppdx}{\sqrt{(1+pp+T^2+2TVp+V^2p^2)}} =$$

$d. \frac{p+TV+V^2p}{\sqrt{(1+pp+T^2+2TVp+V^2p^2)}}$. Eſt vero $Fdx + Gpdx = Fdx + Gdy = dV$, unde erit $\frac{TdV+VpdV}{\sqrt{(1+pp(T+Vp)^2)}} =$

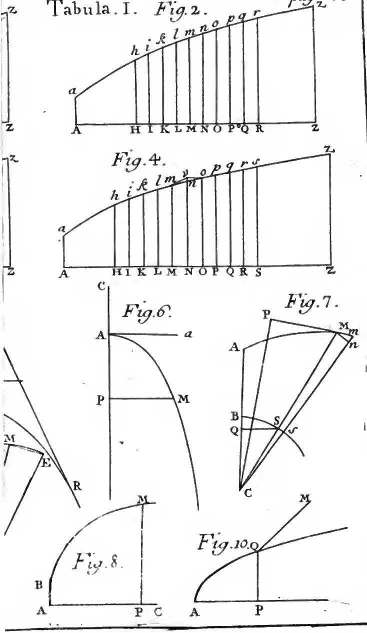
$$d. \frac{p+TV+V^2p}{\sqrt{(1+pp+(T+Vp)^2)}}$$

$$= \left\{ \frac{\begin{array}{l} + dp(1+T^2+V^2) \\ + dT(V-Tp) \\ + dV(T+T^3+3T^2Vp+3TV^2p^2+V^3p^3+2Vp+Vp^3) \end{array}}{(1+pp+(T+Vp)^2)^{3:2}} \right.$$

Æquatione autem ordinata, reſultabit hæc $dp(1+T^2+V^2) + dT(V-Tp) + dV(Vp-Tpp) = 0$, ſeu $dp = \frac{(Tp-V)(dT+pdV)}{1+T^2+V^2}$. Cum vero ſit $p = \frac{dy}{dx}$, erit $dp = \frac{ddy}{dx}$; hincque fiet $dx\,ddy = \frac{(Tdy-Vdx)(dxdT+dydV)}{1+T^2+V^2}$ quæ eſt æquatio differentio-differentialis pro projectione A Q lineæ breviſſimæ in ſuperficie quæſita; ideoque indicat, eam per duo quæque puncta duci poſſe. Æquatio hæc inventa in varias formas tranſmutari poteſt, quæ ſæpius majore commodo uſurpari poterunt. Ac primo quidem expediet eliminari differentialia dT & dV: cum enim ſit $dz = Tdx + Vdy$, erit $ddz = dxdT + dydV + Vddy$; ideoque $dxdT + dydV =$

$ddz - Vddy$, quo valore ſubſtituto prodibit iſta æquatio $dxddy + T^2 dxddy + V^2 dxddy = Tdyddz - Vdxddz - TVdyddy + V^2 dxddy$, ſeu $dxddy + Tdzddy = Tdyddz - Vdxddz$; hincque $ddy : ddz = Tdy - Vdx : dx + Tdz$. Multiplicetur æquatio inventa per dz, ac in primo termino ſcribatur $Tdx + Vdy$ loco dz, erit $Tdx^2 ddy + Vdxdyddy + Tdz^2 ddy = Tdydzddz - Vdxdzddz$. Addatur utrimque $Tdy^2 ddy - Vdxdyddy$, erit $Tddy(dx^2 + dy^2 + dz^2) = (dzddz + dyddy)(Tdy - Vdx)$ ſeu $\frac{dyddy + dzddz}{dx^2 + dy^2 + dz^2} = \frac{Tddy}{Tdy - Vdx} = \frac{Tddz}{dx + Tdz}$. Vel multiplicetur æquatio per dx, ac loco Tdx ſcribatur $dz - Vdy$; obtinebitur $dx^2 ddy + dz^2 ddy - Vdydzddy = dydzddz - Vdy^2 ddz - Vdx^2 ddz$. Addatur utrimque $dy^2 ddy - Vdz^2 ddz$, erit $ddy(dx^2 + dy^2 + dz^2) - Vdz(dyddy + dzddz) = dy(dyddy + dzddz) - Vddz(dx^2 + dy^2 + dz^2)$; ideoque $\frac{dyddy + dzddz}{dx^2 + dy^2 + dz^2} = \frac{ddy + Vddz}{dy + Vdz}$; quæ æquationes omnes in ſequenti expreſſione continentur $\frac{dyddy + dzddz}{dx^2 + dy^2 + dz^2} = \frac{Tddy}{Tdy - Vdx} = \frac{Tddz}{dx + Tdz} = \frac{ddy + Vddz}{dy + Vdz}$. Hic notandum eſt, quia quantitatum T & V differentialia nuſquam occurrunt, perinde eſſe, ſive in T & V contineatur z, ſive minus. Quovis igitur caſu oblato, conveniet eam æquationem aſſumere, quæ facillime integrationem admittat. Veluti ſi ſuperficies propoſita ſit ſolidi rotundi converſione cujuſcunque figuræ circa axem A P nati, erit $yy + zz =$ quadrato functionis ipſius x, quæ ſit $= X$, eſtque applicata illius curvæ genitricis abſciſſæ x reſpondens. Erit itaque $zdz = XdX - ydy$, & $dz = \frac{XdX}{z} - \frac{ydy}{z}$ unde fiet $T = \frac{XdX}{zdx}$, & $V = \frac{-y}{z}$. Sumatur jam, commodi ergo, æquatio in qua T non occur-

Tabula. I.


Fig. 2.
a h i k l m n o p q r z
A H I K L M N O P Q R Z
Fig. 4.
a h i k l m v n o p q r s z
A H I K L M N O P Q R S Z
Fig. 6.
C A a P M
Fig. 7.
P A M m n B S Q s C
M E R
Fig. 8.
M B A P C
Fig. 10.
M Q A P

BIBLIOTECA NAZIONALE VITTORIO EMANUELE
NAPOLI

Digitized by Google

occurrit, hæc $\frac{dyddy + dzddz}{dx^2 + dy^2 + dz^2} = \frac{ddy + Vddz}{dy + Vdz}$, quæ, ob $V = \frac{-y}{z}$, tranſit in hanc $\frac{dyddy + dzddz}{dx^2 + dy^2 + dz^2} = \frac{zddy - yddz}{zdy - ydz}$, cujus integrale eſt $l\sqrt{(dx^2 + dy^2 + dz^2)} = l\frac{zdy - ydz}{b}$, ſeu $zdy - ydz = b\sqrt{(dx^2 + dy^2 + dz^2)}$. Quoniam nunc eſt $z = \sqrt{(X^2 - y^2)}$, ponatur $dX = vdx$, erit $dz = \frac{Xvdx - ydy}{\sqrt{(X^2 - y^2)}}$, & $zdy - ydz = \frac{X^2dy - Xyvdx}{\sqrt{(X^2 - y^2)}}$, & $\sqrt{(dx^2 + dy^2 + dz^2)}$ $= \frac{\sqrt{(X^2dx^2 - y^2dx^2 + X^2dy^2 + X^2v^2dx^2 - 2Xyvdxdy)}}{\sqrt{(X^2 - y^2)}}$. Ergo $X^4dy^2 - 2X^3yvdxdy + X^2y^2v^2dx^2 = bbX^2dx^2 - bby^2dx^2 + bbX^2dy^2 + b^2X^2v^2dx^2 - 2b^2Xyvdxdy$, ſeu dy^2 $= \frac{2(b^2 - X^2)Xyvdxdy + X^2y^2v^2dx^2 - b^2X^2dx^2 + b^2y^2dx^2 - b^2X^2v^2dx}{X^2(bb - XX)}$, quæ, extracta radice, præbet $dy = \frac{yvdx}{X} \pm \frac{bdx\sqrt{(1+vv)(yy - XX)}}{X\sqrt{(bb - XX)}}$. Quod ſi ponatur $y = Xt$, ut ſit $dy = Xdt + tvdx$, fiet $\frac{dt}{\sqrt{(tt - 1)}} = \frac{bdx\sqrt{(1 + vv)}}{X\sqrt{(bb - XX)}}$; in qua æquatione, quia X & v ſunt functiones ipſius x, variabiles t & x a ſe invicem ſunt ſeparatæ.

Exemplum V.

12. *Super axe* APN *conſtruere curvam* AM *ejuſmodi, ut, abſciſſa per normalem* MN *area* ANM *data magnitudinis, arcus* AM *ſit minimus.* Fig. 11.

Quia, pro definita areæ AMN magnitudine, arcus AM minimus eſſe debet, ponatur area AMN $= ax$, poſitoque $x = a$, quo caſu area AMN fit $= aa$, fiat arcus AM minimus. Ponatur porro applicata orthogonalis MP $= y$, abſciſſa AP $= t$, & ſubnormalis PN $= u$; erit $ax = \int y\,dt + \frac{1}{2}uy$, & $u = \frac{ydy}{dt}$: elementum vero arcus AM erit $=$

$\frac{dy\sqrt{(yy+uu)}}{u}$. Porro cum ſit $a\,dx = y\,dt + \frac{1}{2}(u\,dy + y\,du)$ & $dt = \frac{y\,dy}{u}$, erit $au\,dx = yy\,dy + \frac{1}{2}uu\,dy + \frac{1}{2}yu\,du$, & $du = \frac{2a\,dx}{y} - \frac{2y\,dy}{u} - \frac{u\,dy}{y}$. Jam ponatur $dy = p\,dx$, minimum eſſe debebit $\int \frac{p\,dx\sqrt{(yy+uu)}}{u}$, atque u eſt quantitas cujus valor ex hac æquatione $du = dx\left(\frac{2a}{y} - \frac{2yp}{u} - \frac{up}{y}\right)$ definiri debet. Pertinet itaque hæc quæſtio ad Caſum quintum; cum quo ſi comparatio inſtituatur, fit $u = \Pi$ & $Z = \frac{p\sqrt{(yy+\Pi^2)}}{\Pi}$; unde $L = \frac{-pyy}{\Pi^2\sqrt{(yy+\Pi^2)}}$; $M = 0$, $N = \frac{yp}{\Pi\sqrt{(yy+\Pi^2)}}$, & $P = \frac{\sqrt{(yy+\Pi\Pi)}}{\Pi}$. Deinde cum ſit $\Pi = \int dx\left(\frac{2a}{y} - \frac{2yp}{\Pi} - \frac{\Pi p}{y}\right)$, fit $[Z] = \frac{2a}{y} - \frac{2yp}{\Pi} - \frac{\Pi p}{y}$, & differentiando erit $[L] = \frac{2yp}{\Pi^2} - \frac{p}{y}$; $[M] = 0$, $[N] = \frac{-2a}{yy} - \frac{2p}{\Pi} + \frac{\Pi p}{yy}$ & $[P] = \frac{-2y}{\Pi} - \frac{\Pi}{y}$. Jam erit $\int[L]dx = \int\frac{2y\,dy}{\Pi^2} - ly$, & $e^{\int[L]dx} = \frac{e^{\int 2y\,dy:\Pi\Pi}}{y}$; at eſt $L\,dx = \frac{-yy\,dy}{\Pi^2\sqrt{(yy+\Pi\Pi)}}$; unde fiet $\int e^{\int[L]dx}L\,dx = -\int\frac{e^{\int 2y\,dy:\Pi\Pi}\,y\,dy}{\Pi^2\sqrt{(yy+\Pi^2)}}$, cujus valor poſito $x = a$, fiat $= H$, ſitque $V = e^{-\int 2y\,dy:\Pi\Pi}\,y\left(H + \int\frac{e^{\int 2y\,dy:\Pi\Pi}\,y\,dy}{\Pi^2\sqrt{(yy+\Pi^2)}}\right)$. His præparatis, erit æquatio ſatisfaciens $(N+[N]V)dx = d.(P+[P]V)$, ſive ſubſtitutionibus factis, $\frac{y\,dy}{\Pi\sqrt{(yy+\Pi\Pi)}} - \frac{2aV\,dx}{yy} - \frac{2V\,dy}{\Pi} - \frac{\Pi V\,dy}{yy} = d.\left(\frac{\sqrt{(yy+\Pi\Pi)}}{\Pi} - \frac{2Vy}{\Pi} - \frac{\Pi V}{y}\right)$. At eſt $2a\,dx = y\,d\Pi$

+

Digitized by Google

$+ \frac{2yydy}{\Pi} + \Pi dy$; unde erit $\frac{ydy}{\Pi\sqrt{(yy+\Pi^2)}} - \frac{Vd\Pi}{y} - \frac{4Vdy}{\Pi}$ $= d.\left(\frac{\sqrt{(yy+\Pi^2)}}{\Pi} - \frac{2Vy}{\Pi} - \frac{\Pi V}{y}\right) = \frac{ydy}{\Pi\sqrt{(yy+\Pi^2)}} -$ $\frac{yyd\Pi}{\Pi^2\sqrt{(yy+\Pi^2)}} - \frac{2Vdy}{\Pi} - \frac{2ydV}{\Pi} + \frac{2Vyd\Pi}{\Pi^2} - \frac{\Pi dV}{y} - \frac{Vd\Pi}{y}$ $+ \frac{\Pi Vdy}{yy})$; hincque $\frac{yyd\Pi}{\Pi^2\sqrt{(y^2+\Pi^2)}} - \frac{2Vdy}{\Pi} + \frac{2ydV}{\Pi} - \frac{2Vyd\Pi}{\Pi^2}$ $+ \frac{\Pi dV}{y} - \frac{\Pi Vdy}{yy} = 0$. Verum, eſt generaliter $dV = -Ldx$ $- V[L]dx$; unde erit $dV = \frac{yydy}{\Pi^2 V(yy+\Pi\Pi)} - \frac{2Vydy}{\Pi^2}$ $+ \frac{Vdy}{y}$; hincque $\frac{yyd\Pi}{\Pi^2\sqrt{(yy+\Pi\Pi)}} = \frac{dVd\Pi}{dy} + \frac{2Vyd\Pi}{\Pi^2} -$ $\frac{Vd\Pi}{y}$; quo ſubſtituto oritur $\frac{dVd\Pi}{dy} - \frac{2Vdy}{\Pi} + \frac{2ydV}{\Pi} + \frac{\Pi dV}{y}$ $- \frac{Vd\Pi}{y} - \frac{\Pi Vdy}{yy} = 0$; hoc eſt $dV\left(\frac{d\Pi}{dy} + \frac{2y}{\Pi} + \frac{\Pi}{y}\right)$ $= V\left(\frac{d\Pi}{y} + \frac{2dy}{\Pi} + \frac{\Pi dy}{yy}\right) = \frac{ydV}{dy}\left(\frac{d\Pi}{y} + \frac{2dy}{\Pi} + \frac{\Pi dy}{yy}\right)$; quæ æquatio, cum ſit diviſibilis per $\frac{d\Pi}{y} + \frac{2dy}{\Pi} + \frac{\Pi dy}{yy}$, duplicem dat ſolutionem. Quarum prima erit $\frac{dV}{V} = \frac{dy}{y}$, quæ præbet $V = cy$: quoniam vero V evaneſcere debet in caſu minimi, eodem caſu erit $y = 0$; ſcilicet poſito $x = a$ fiet $y = 0$. Cum nunc ſit $V = cy$, facta ſubſtitutione in æquatione $dV = \frac{yydy}{\Pi^2\sqrt{(y^2+\Pi^2)}} - \frac{2Vydy}{\Pi^2} + \frac{Vdy}{y}$, erit $\frac{yydy}{\Pi^2\sqrt{(y^2+\Pi^2)}}$ $= \frac{2cyydy}{\Pi^2}$; hincque vel $y = 0$, vel $dy = 0$, quo caſu prodit linea recta axi parallela; vel $\Pi = \infty$, quo caſu prodit linea recta ad axem normalis: vel etiam $\sqrt{(yy+\Pi\Pi)} = MN =$ *Conſt.* quæ æquatio dat Circulum; atque integer ſemicirculus, ob $y = 0$ in caſu minimi, quæſito ſatisfaciet. Secunda ſolutio prodit ex diviſore $\frac{d\Pi}{y} + \frac{2dy}{\Pi} + \frac{\Pi dy}{yy} = 0$, ſeu $\Pi d\Pi + \frac{\Pi\Pi dy}{y}$

+

Digitized by Google

$+ 2ydy = o$, quæ multiplicata per yy, fit $yy\pi d\pi + \pi\pi ydy + 2y^3dy = o$, cujus integrale eſt $\pi^2y^2 + y^4 = C$, hincque $\pi = \frac{\sqrt{(b^4 - y^4)}}{y}$; quæ æquatio, quia non pendet ab V, pro quocunque valore ipſius x ſatisfaciet. Erit autem introducta abſciſſa $AP = t$, ob $u = \pi = \frac{ydy}{dt}$, iſta æquatio $\frac{ydy}{dt} = \frac{\sqrt{(b^4 - y^4)}}{y}$, unde $dt = \frac{yydy}{\sqrt{(b^4 - y^4)}}$, ex qua æquatione intelligitur Elaſticam rectangulam quæſito ſatisfacere; ita ut pro area ANM inter normales AN & MN arcus curvæ AM ſit breviſſimus. Hæc autem curva per data duo puncta, ſiquidem axis AP ſit poſitione datus, deſcribi poteſt.

SCHOLION II.

13. Ex his Exemplis eximius uſus, quem habet noſtra Methodus in Problematis etiam diverſi generis reſolvendis, abunde patet; inprimis autem ultimum Exemplum nonnullas notatu maxime dignas ſuppeditat circumſtantias, ex quibus natura ſolutionis illuſtrari poterit. Quoniam enim duplex æquatio ob factores duos nata eſt, duplex quoque ſolutio prodiit; quarum prior lineam ſatisfacientem abſolute determinat, ita ut ea per data duo puncta duci nequeat: dat enim vel lineam rectam, vel ſemicirculum. Linea recta duplici modo quæſtionem ſolvit, dum eſt vel normalis ad axem AP, vel eidem parallela; & quemadmodum utraque ſatisfaciat manifeſtum eſt: nam in ea, quæ eſt normalis ad axem, portio quæ cum axe & normali datum ſpatium comprehendit perpetuo eſt infinite parva, ideoque revera minima: altera recta axi parallela aliquanto latius patet, cum ea per datum punctum duci poſſit; & quia ipſæ applicatæ ad eam ſunt normales, ac ſpatium abſciſſum ſit ut ipſa abſciſſa, ejus reſpectu linea illa recta utique erit breviſſima. Semicirculus deinde, qui ex prima ſolutione prodiit, ita abſolute ſatisfacit, ut, propoſita ſpatii abſcindendi quantitate, ipſe ſemircirculus determinetur,

Digitized by Google

netur, ejus enim area eſſe debet $= aa$. Secunda autem ſolutio, quæ curvam Elaſticam rectangulam præbuit, latius patet: nam per data duo quæcunque puncta ejuſmodi curva traduci poteſt, eaque, inter omnes alias curvas per eadem puncta tranſeuntes, hac gaudebit prærogativa, ut ſi, in omnibus curvis, per normales, areæ æquales abſcindantur, arcus Elaſticæ futurus ſit omnium minimus. His igitur expoſitis pergamus ad uſum Methodi traditæ oſtendendum, in iis maximi minimive inveſtigationibus, in quibus maximi minimive formula non eſt talis expreſſio integralis ſimplex $\int Z dx$, qualem formam hactenus perpetuo tractavimus; verum eſt compoſita ex duabus pluribuſve hujuſmodi formulis quomodocunque. Ac primo quidem, ſi maximum minimumve eſſe debeat aggregatum duarum pluriumve formularum integralium, puta $\int Z dx + \int Y dx - \int X dx$, operatio nulla difficultate laborat: quia enim formula maximi minimive eſt $\int dx\,(Z + Y - X)$, hæc tanquam ſimplex formula integralis tractari, ejuſque valor differentialis aſſignari poterit. Operatio autem eo redibit, ut pro ſingulis formulis $\int Z dx$, $\int Y dx$ & $\int X dx$, earum valores differentiales quærantur; earumque loco in formula $\int Z dx + \int Y dx - \int X dx$ ſubſtituantur; & quod oritur nihilo æquale ponatur: ſicque habebitur æquatio quæſito ſatisfaciens.

PROPOSITIO II. PROBLEMA.

14. *Invenire æquationem inter* x *&* y, *ut, poſito* $x = a$, *fiat hac expreſſio* $\int Z dx \times \int Y dx$, *quæ eſt productum ex duabus formulis integralibus* $\int Z dx$ *&* $\int Y dx$, *maximum vel minimum.*

SOLUTIO.

Ponamus iſtam æquationem inter x & y jam eſſe inventam, foreque ex ea, poſito $x = a$, valorem Formulæ $\int Z dx = A$, & $\int Y dx = B$; erunt hæ quantitates A & B conſtantes; atque earum productum AB maximum vel minimum. Jam ponatur apud valorem indefinitum x variabilem y augeri particula

$n\nu$, ex ea utraque quantitas A & B incrementum accipiet, unaquæque ſcilicet augebitur valore differentiali ex præcedentibus definiendo. Sit igitur dA valor differentialis ipſius A, qui reſpondet formulæ integrali $\int Z\,dx$, poſito $x = a$, ſimilique modo ſit dB valor differentialis ipſius B oriundus ex formula $\int T\,dx$, poſito $x = a$. Cum ergo, ex adjecta particula $n\nu$ variabili y, abeat A in $A + dA$, & B in $B + dB$, productum AB tranſmutabitur in $AB + A\,dB + B\,dA + dA\,dB$; quare cum AB eſſe debeat maximum vel minimum, oportebit eſſe $AB = AB + A\,dB + B\,dA + dA\,dB$. Ideoque $0 = A\,dB + B\,dA$, ob evaneſcentem terminum $dA\,dB$ præ reliquis. Ex his itaque oritur ſequens Problematis ſolutio; Quæratur formulæ $\int Z\,dx$ valor differentialis qui ſit dA, ſitque A valor formulæ $\int Z\,dx$, quem obtinet poſito $x = a$. Deinde quæratur formulæ $\int T\,dx$ valor differentialis, qui ſit dB, ac B denotet valorem formulæ $\int T\,dx$, quem recipit poſito $x = a$: quibus factis habebitur iſta æquatio $0 = A\,dB + B\,dA$, in qua relatio ſatisfaciens inter x & y continebitur. *Q. E. I.*

COROLL. I.

15. Quanquam in æquatione $0 = A\,dB + B\,dA$ inſunt quantitates conſtantes A & B, tamen eæ non ſunt arbitrariæ, ſed utraque per ipſam hanc æquationem definietur. Scilicet ſi ex hac æquatione eliciantur valores $\int Z\,dx$ & $\int T\,dx$, ponaturque $x = a$, prodire debent illæ quantitates A & B; unde hæ determinabuntur per a, & per reliquas conſtantes arbitrarias quæ per integrationem ingredientur.

COROLL. II.

16. Si Z & T fuerint functiones determinatæ quantitatum x, y, p, q, r, &c. tum valores differentiales dA & dB non pendebunt ab a; interim tamen quantitas a ingreditur in æquationem $0 = A\,dB + B\,dA$: ex quo curva inventa, tantum pro definito abſciſſæ x valore $x = a$, quæſito ſatisfaciet.

Co-

Digitized by Google

COROLL. III.

17. Ex æquatione autem $o = AdB + BdA$ particula $n\nu$ omnino egredictur: nam quia uterque valor differentialis dA & dB per $n\nu$ multiplicatus prodiit, iterum $n\nu$ per diviſionem exterminabitur: hocque modo æquatio inter x & y atque conſtantes naſcetur, qua Problemati ſatisfiet.

SCHOLION I.

18. Neminem hic forma æquationis $o = AdB + BdA$ inventæ offendat, eo quod ſpeciem formulæ differentialis definitæ præ ſe ferat, neque hinc etiam quiſquam concludat æquationis $o = AdB + BdA$ integralem ſumi poſſe hanc, *Conſt.* $= AB$. Jam enim ſignificationes explicavimus, quas tribuimus cum litteris A & B, tum etiam formis differentialibus dA & dB: ex quo intelligere licet, vulgarem notandi modum hic non locum habere. Ideo autem hunc notandi modum, etſi a conſueto diſſentientem, hic adhibere viſum eſt, ut nexus æquationis $o = AdB + BdA$ cum formula maximi minimive $\int Zdx . \int Tdx$ melius perſpiciatur. Cum enim maximum minimumve reſpondere debeat valori $x = a$; ponamus hoc caſu abire $\int Zdx$ in A & $\int Tdx$ in B; quo facto, maximum minimumve erit AB. Hinc autem ſponte naſcitur æquatio inventa $o = AdB + BdA$, ſiquidem AB, litteris A & B tanquam variabilibus ſpectatis, differentietur. Quod cum fuerit factum, in memoriam revocari oportet, pro differentialibus dA & dB accipiendos eſſe valores differentiales eos, qui conveniunt formulis integralibus $\int Zdx$ & $\int Tdx$, ex quibus ipſæ quantitates A & B conſtantes prodiere. Hunc nexum ideo annotaſſe juvabit, quod infra eundem ad quemcunque compoſitionis modum, quo formula maximi minimive ex formulis integralibus compoſita fuerit, æque patere; ſimilique modo ex ipſa maximi minimive expreſſione per differentiationem æquationem quæſitam obtineri oſtendemus.

Digitized by Google

EXEMPLUM I.

19. *Invenire æquationem inter* x *&* y, *ut*, *poſito* $x = a$, *fiat iſta expreſſio* $\int y\,dx \times \int x\,dy$ *maximum.*

Fiat $\int y\,dx = A$, & $\int x\,dy = B$, poſito $x = a$, & quærantur formularum $\int y\,dx$ & $\int x\,dy$, ſeu $\int x p\,dx$, valores differentiales: ac formulæ $\int y\,dx$ valor differentialis eſt $nv.\,dx.\,1$, formulæ autem $\int x\,dy$, ſeu $\int x p\,dx$, eſt $nv.\,dx\,(-\frac{1}{dx}\;d.\,x) = -nv.\,dx$. Erit ergo $dA = nv.\,dx$, & $dB = -nv.\,dx$: unde æquatio $o = AdB + BdA$ abibit in hanc $o = -A.\;nv.\,dx + B.\;nv.\,dx$, ſeu $A = B$. Quæſito ergo omnes æquationes inter x & y æque ſatisfaciunt, dummodo, caſu $x = a$, fuerit $\int y\,dx = \int x\,dy$; hoc eſt area curvæ $= \frac{1}{2} xy$.

EXEMPLUM II.

20. *Invenire æquationem inter* x *&* y, *ut*, *caſu* $x = a$, *fiat minimum hac expreſſio* $\int y\,dx \times \int dx \surd(1+pp)$.

Caſu $x = a$, fiat $\int y\,dx = A$, & $\int dx \surd(1+pp) = B$. Porro ſumendis valoribus differentialibus erit $dA = nv.\,dx.\,1$, & $dB = nv.\,dx\,(-\frac{1}{dx}\;d.\,\frac{p}{\surd(1+pp)}) = -nv.\,d.\,\frac{p}{\surd(1+pp)}$. Hinc prodit ſequens æquatio $o = -A.\,nv.\,d\,\frac{p}{\surd(1+pp)} + B.\,nv.\,dx$, ſeu $B\,dx = A\,d.\,\frac{p}{\surd(1+pp)}$. Quæ integrata dat $x + b = \frac{Ap}{B\surd(1+pp)}$, ubi $\frac{A}{B}$ denotat rationem, quam tenet $\int y\,dx$ ad $\int dx \surd(1+pp)$ tum cum fit $x = a$. Sit brevitatis gratia $\frac{A}{B} = c$, erit $(x+b)\surd(1+pp) = cp$, & $p = \frac{x+b}{\surd(cc-(x+b)^2)} = \frac{dy}{dx}$. Integrata ergo hac æquatione,

tione; reſultabit $y = f \pm \sqrt{(cc - (x+b)^2)}$, ita ut ſit $(y-f)^2 + (x+b)^2 = c^2$, unde patet curvam ſatisfacientem eſſe Circulum, radio c deſcriptum, axe ubicunque accepto. Hujuſmodi vero Circuli non quivis arcus ſatisfaciet, verum is tantum qui per c radium Circuli multiplicatus producit aream; eſt enim $A = Bc$. Ergo vel radius Circuli c pro lubitu accipi poteſt, ex eoque definietur illa abſciſſæ x magnitudo determinata a; vel ſi a detur, ut ponimus, inde viciſſim radius c determinabitur. Perſpicuum autem eſt arcum Circuli, qui ſatisfacit, convexitate ſua axem reſpicere debere; hoc enim caſu area fit minor, ideoque productum ex area in arcum minimum.

EXEMPLUM III.

21. *Invenire curvam, in qua, pro data abſciſſa* $x = a$, *minimum fiat hac expreſſio* $\int yx\,dx \times \int x\,dx \sqrt{(1+pp)}$.

Poſito $x = a$, fiat $\int yx\,dx = A$, & $\int x\,dx \sqrt{(1+pp)} = B$. Erit autem $dA = ny.\,dx.\,x$ & $dB = -m.\,dx.\,\frac{1}{dx}\,d.\,\frac{xp}{\sqrt{(1+pp)}}$; unde obtinebitur iſta æquatio $Bx\,dx = A\,d.\,\frac{px}{\sqrt{(1+pp)}}$, quæ integrata dat $xx \pm bb = \frac{2Apx}{B\sqrt{(1+pp)}} = \frac{2cpx}{\sqrt{(1+pp)}}$, poſito $\frac{A}{B} = c$. Hinc $p = \frac{xx \pm bb}{\sqrt{(4ccxx - (xx \pm bb)^2)}} = \frac{dy}{dx}$, ideoque pro curva habebitur hæc æquatio, $y = \int \frac{(xx \pm bb)\,dx}{\sqrt{(4ccxx - (xx \pm bb)^2)}}$. De qua notandum eſt, ſi fiat $b = 0$, tum prodire æquationem pro Circulo $y = \int \frac{x\,dx}{\sqrt{(4cc - xx)}}$ cujus radius ſit $2c$.

SCHOLION II.

22. Eadem hæc Exempla omnia quoque reſolvi poſſunt per Methodum ſupra jam traditam; quare cum utraque via eadem ſolutio obtineatur, juvabit ſolutionem per alteram viam uno Exemplo exhiberi. Sumamus igitur tertium Exemplum, in quo maximi minimive formula $\int yx\,dx \times \int x\,dx\,\sqrt{(1+pp)}$, differentiando iterumque integrando per partes, reducitur ad hanc formam $\int yx\,dx \int x\,dx\,\sqrt{(1+pp)} + \int x\,dx\,\sqrt{(1+pp)} \int yx\,dx$; cujus utrumque membrum in Caſu ſecundo ſupra §. 7. expoſito continetur. Quæratur itaque utriuſque valor differentialis, eorum enim ſumma, poſita $=0$, dabit æquationem pro curva quæſita. Formula autem $\int yx\,dx \int x\,dx\,\sqrt{(1+pp)}$ cum Caſu ſecundo collata, dabit $\Pi = \int x\,dx\,\sqrt{(1+pp)}$ & $Z = yx\Pi$; unde fit $L = yx$; $M = y\Pi$, $N = x\Pi$, $P = 0$, &c. Deinde erit $[Z] = x\sqrt{(1+pp)}$; indeque $[M] = \sqrt{(1+pp)}$, $[N] = 0$, & $[P] = \frac{xp}{\sqrt{(1+pp)}}$. Porro eſt $\int L\,dx = \int yx\,dx$, cujus valor, poſito $x = a$, quem generaliter poſuimus H, hic in ſolutione Exempli eſt A; ita ut ſit $V = A - \int yx\,dx$. Quare hujus formulæ valor differentialis erit $= n\nu . dx \,(x\Pi - \frac{1}{dx} d. \frac{xp(A - \int yx\,dx))}{\sqrt{(1+pp)}} = n\nu . dx\,(x\int x\,dx\,\sqrt{(1+pp)} - \frac{A}{dx} d. \frac{xp}{\sqrt{(1+pp)}} + \frac{1}{dx} d. \frac{xp\int yx\,dx}{\sqrt{(1+pp)}})$. Altera formula $\int x\,dx\,\sqrt{(1+pp)} \int yx\,dx$, cum Caſu ſecundo §. 7. collata, dat $\Pi = \int yx\,dx$ & $Z = x\Pi\sqrt{(1+pp)}$, unde erit $L = x\sqrt{(1+pp)}$, $M = \Pi\sqrt{(1+pp)}$, $N = 0$, & $P = \frac{x\Pi p}{\sqrt{(1+pp)}}$; hincque $\int L\,dx = \int x\,dx\,\sqrt{(1+pp)}$: quare cum H ſit valor ipſius $\int L\,dx$, poſito $x = a$, erit $H = B$, & $V = B - \int x\,dx\,\sqrt{(1+pp)}$. Porro eſt $[Z] = yx$, hincque $[M] = y$, $[N] = x$, & $[P] = 0$. Ex his prodit valor differentialis $= n\nu . dx\,(Bx - x\int x\,dx\,\sqrt{(1+pp)} - \frac{1}{dx} \times$

d.

Digitized by Google

$\times d. \frac{xp\int yx\,dx}{\sqrt{(1+pp)}}$). His igitur valoribus differentialibus ambobus additis, emerget hujus expreſſionis compoſitæ $\int yx\,dx \int x\,dx \sqrt{(1+pp)} + \int x\,dx \sqrt{(1+pp)} \int yx\,dx$, ſeu hujus $\int yx\,dx \times \int x\,dx \sqrt{(1+pp)}$, quæ in Exemplo erat propoſita, valor differentialis $= nv.\ dx\ (Bx - \frac{A}{dx}\ d.\ \frac{xp}{\sqrt{(1+pp)}})$, ex quo pro curva æquatio erit hæc $Bx\,dx = A\,d.\ \frac{xp}{\sqrt{(1+pp)}}$, quam eandem in ſolutione Exempli invenimus. Similis autem conſenſus in genere deprehendetur, ſi quis expreſſionem $\int Z\,dx \times \int T\,dx$ eodem modo tractare voluerit.

PROPOSITIO III. PROBLEMA.

23. *Invenire æquationem inter* x *&* y *ejus conditionis, ut, poſita* $x = a$, *iſta fractio* $\frac{\int Z\,dx}{\int Y\,dx}$ *obtineat maximum minimumve valorem: exiſtentibus* Z *&* Y *functionibus quibuſcunque ipſarum* x, y, p, q, r, *&c. ſive determinatis, ſive indeterminatis.*

SOLUTIO.

Caſu quo fit $x = a$, fit $\int Z\,dx = A$, atque $\int T\,dx = B$: eritque $\frac{A}{B}$ maximum vel minimum, ſiquidem relatio inter x & y recte fuerit aſſignata. Erit igitur fractio $\frac{A}{B}$ æqualis eidem huic fractioni $\frac{\int Z\,dx}{\int T\,dx}$, caſu quo $x = a$, ſi alicubi una applicata y augeatur particula $n\nu$. Tum vero fiet $\int Z\,dx$ æqualis ipſi A, una cum valore differentiali formulæ $\int Z\,dx$, qui ſit $= dA$; ſimilique modo $\int T\,dx$ abibit in B auctum valore differentiali formulæ $\int T\,dx$, qui ſit $= dB$; ſicque ex adjecta particula $n\nu$ ad applicatam y, caſu quo $x = a$, tranſibit fractio $\frac{\int Z\,dx}{\int T\,dx}$ in hanc $\frac{A+dA}{B+dB}$; quæ æqualis eſſe debet fractioni

A

Digitized by Google

$\frac{A}{B}$; unde naſcitur iſta æquatio $BdA = AdB$; quæ præbebit æquationem inter x & y quæſitam. *Q. E. I.*

Coroll. I.

24. Ad hanc igitur æquationem inter x & y inveniendam, effici debet ut valores differentiales ipſarum $\int Zdx$ & $\int Tdx$ proportionales fiant ipſis harum formularum valoribus, quos obtinent poſito $x = a$.

Coroll. II.

25. Quanquam, in hac æquatione inventa $BdA = AdB$, duæ ineſſe videantur conſtantes incognitæ A & B, tamen ambas in unam compingere licet. Poſito enim $\frac{A}{B} = C$, erit $dA = CdB$; inventaque æquatione, ex valore a loco x ſubſtituto determinabitur valor ipſius C.

Scholion.

26. Si hujus & præcedentis Problematis ſolutiones inter ſe conferantur, ingens in iis deprehendetur conſenſus. Nam ſi maximum minimumve eſſe debeat factum $\int Zdx \times \int Tdx$, orta eſt iſta æquatio $o = AdB + BdA$; ſin autem quotus $\frac{\int Zdx}{\int ydx}$ debeat eſſe vel maximus vel minimus, inventa eſt iſta æquatio $o = AdB - BdA$; utroque autem caſu litteræ A, B & dA, dB eoſdem retinent valores. Quare cum A & B ſint quantitates conſtantes, ambæ æquationes tantum ratione ſigni conſtantis differunt; poſito enim $\frac{A}{B} = C$, priore caſu habetur $dA = -CdB$, poſteriore vero $dA = +CdB$. Ex quo pro utroque caſu etiam eadem fere prodibit ſolutio; quia totum diſcrimen tantum in ſigno quantitatis conſtantis C ſitum erit. Quod ſi ergo æquatio inter x & y fuerit inventa, quæ conti-

Digitized by Google

contineat, pro $x = a$, factum $\int Z dx \times \int Y dx$ maximum vel minimum; eadem æquatio, levi adhibita mutatione, ſimul continebit quotum $\frac{\int Z dx}{\int Y dx}$ maximum vel minimum. Perſpicuum autem eſt, ſive $\frac{\int Z dx}{\int Y dx}$ debeat eſſe maximum vel minimum, ſive $\frac{\int Y dx}{\int Z dx}$, utroque caſu eandem plane eſſe prodituram æquationem. Hanc vero convenientiam ipſa rei natura poſtulat: nam ſi $\frac{\int Z dx}{\int Y dx}$ eſt maximum, tum eo ipſo erit $\frac{\int Y dx}{\int Z dx}$ minimum & viciſſim; unde utrique quæſtioni eandem ſolutionem ſatisfacere neceſſe eſt. Cæterum hunc quoque nexum obſervaſſe juvabit inter maximi minimive formulam $\frac{\int Z dx}{\int Y dx}$, quæ, poſito $x = a$, abit in $\frac{A}{B}$, & inter æquationem inventam $BdA - AdB = 0$: hæc enim æquatio oritur ex differentiatione formulæ $\frac{A}{B}$, ponendo ejus differentiale $= 0$; iſtiuſmodi autem nexum perpetuo locum habere in ſequente Propoſitione demonſtrabimus.

EXEMPLUM I.

27. *Invenire curvam, cujus area coordinatis orthogonalibus abſciſſa ad arcum curvæ maximam teneat rationem, ſi abſciſſæ datus valor* a *tribuatur.*

Poſita curvæ quæſitæ abſciſſa $= x$, applicata $= y$; erit area $= \int y dx$, & arcus $= \int dx \sqrt{(1+pp)}$; poſito $dy = p dx$: maximum ergo eſſe debet $\frac{\int y dx}{\int dx \sqrt{(1+pp)}}$, caſu quo ponitur $x = a$. Sit igitur, caſu $x = a$, valor formulæ $\int y dx$, ſeu area $= A$, & $\int dx \sqrt{(1+pp)}$ ſeu arcus abſciſſæ a reſpondens $= B$. Deinde formulæ $\int y dx$ valor differentialis dA erit $= nv.\, dx$.

1, & formulæ $\int dx \surd(1+pp)$ ſeu $dB = nv.\, dx\,(-\frac{1}{dx}\, d.\, \frac{p}{\surd(1+pp)}) = -nv.\, d.\, \frac{p}{\surd(1+pp)}$. Quibus valoribus in æquatione $BdA = AdB$ ſubſtitutis, prodibit pro curva quæſita ſequens æquatio: $Bdx = -A d.\, \frac{p}{\surd(1+pp)}$. Ponatur $\frac{A}{B} = c$, ita ut, pro abſciſſa $x = a$, area curvæ fiat æqualis producto ex arcu in hanc conſtantem c. Erit ergo $dx = -c d.\, \frac{p}{\surd(1+pp)}$, & integrando $x = b - \frac{cp}{\surd(1+pp)}$, ſeu $cp = (b-x)\surd(1+pp)$, hincque $p = \frac{b-x}{\surd(c^2-(b-x)^2)} = \frac{dy}{dx}$. Erit ergo $y = \ldots \int \frac{(b-x)dx}{\surd(c^2-(b-x)^2)} = f \pm \surd(c^2-(b-x)^2)$ ſeu $(y-f)^2 + (b-x)^2 = cc$; unde conſtat curvam quæſitam eſſe Circulum radio c deſcriptum, ad rectam quamcunque tanquam axem relatum. Hujus autem Circuli ea tantum portio quæſito ſatisfacit, quæ reſpondet abſciſſæ $= a$, a quo valore pendet c, ita ut ſumpta abſciſſa $= a$, area æqualis fiat producto ex arcu in radium Circuli multiplicato. Quod ſi ergo viciſſim radius c detur, tanta in axe abſciſſa abſcindi debet, ut arcus per radium multiplicatus præbeat aream. Infinitis igitur modis quæſito ſatisfieri poteſt; quæſtio autem erit determinata, ſi duo præſcribantur puncta, per quæ curva quæſita ſit tranſeunda. Sumamus igitur radium c tanquam cognitum, eoque deſcribamus Circulum BMD centro C. Porro ſumatur linea quæcunque APD pro axe, in eaque A pro origine abſciſſarum. Hoc jam facto, quæſtioni ſatisfiet ſi applicata PM tantum ſpatium ABMP abſcindatur, ut id ſit æquale producto ex arcu BM in radium Circuli BC. Quia autem ſector BCM eſt $= \frac{1}{2}$ BM. BC, oportet aream ABMP eſſe duplo majorem ſectore BCM. Apparet autem, ſumto pro lubitu,

Fig. 12.

bitu, cum axe, tum ejus initio, ſæpe-numero conditionem præſcriptam nequidem impleri poſſe. Nam ſi axis AD per centrum tranſeat, tum area ABMP perpetuo minor erit quam duplum ſectoris BCM; niſi, arcu BM infinite parvo, prima applicata BA ſimul per centrum tranſeat: ſin autem axis AD ſupra centrum tranſiret, tum nullo modo conditioni inventæ ſatisfieri poteſt. Quare neceſſe eſt, ut axis AD infra centrum C ducatur, qua de re multæ egregiæ obſervationes geometricæ fieri poſſent, ſi ratio inſtituti id permitteret. Cæterum ſi hæc Solutio cum Exemplo ſecundo præced. Prop. §. 20 comparetur; apparebit eandem prorſus æquationem eſſe inventam, ſive $\int y\,dx \times \int dx\sqrt{(1+pp)}$ debeat eſſe minimum, ſive $\frac{-\int y\,dx}{\int dx\sqrt{(1+pp)}}$ maximum. Diſcrimen tamen in hoc conſiſtit; quod radius Circuli $c = \frac{A}{B}$ altero caſu affirmative, altero negative debeat accipi. Scilicet ſi $\int y\,dx \times \int dx\sqrt{(1+pp)}$ debeat eſſe minimum, arcus BM convexitate ſua ſpatium ABMP; altero autem caſu, concavitate claudere debet.

EXEMPLUM II.

28. *Intra datum angulum* ACM, *curvam* AM *conſtruere ita comparatam, ut area* ACM *per arcum* AM *diviſa ſit omnium maxima.* Fig. 7.

Ponatur angulus ACM, ſeu arcus circuli BS radio CB $= 1$ deſcriptus $= x$, qui in caſu propoſito fiat $= a$, quo $\frac{ACM}{AM}$ fieri debet maximum. Ponatur porro $CM = y$, ſitque $dy = p\,dx$, erit $Mn = y\,dx$, & area $ACM = \frac{1}{2}\int yy\,dx$: arcus autem AM reperitur $= \int dx\sqrt{(yy+pp)}$: unde hæc fractio $\frac{\int yy\,dx}{2\int dx\sqrt{(yy+pp)}}$, ſeu ejus duplum $\frac{\int yy\,dx}{\int dx\sqrt{(yy+pp)}}$ debebit eſſe maximum. Sit, caſu quo $x = a$

 eſt,

Digitized by Google

eſt, $\int yydx = A$, & $\int dx\sqrt{(yy+pp)} = B$; erit, ſi $x = a$, area ACM $= \frac{1}{2}A$, & arcus AM $= B$. Jam formulæ $\int yydx = A$ valor differentialis dA eſt $= nv.\, dx.\, 2y$, & formulæ $\int dx\sqrt{(yy+pp)}$ valor differentialis dB eſt $= nv.\, dx\left(\frac{y}{\sqrt{(yy+pp)}} - \frac{1}{dx}\, d.\, \frac{p}{\sqrt{(yy+pp)}}\right)$. Quare cum generaliter invenerimus pro curva hanc æquationem $BdA = AdB$, erit diviſione per nv inſtituta, $2Bydx = \frac{Aydy}{\sqrt{(yy+pp)}} - Ad.\, \frac{p}{\sqrt{(yy+pp)}}$. Multiplicetur ea per p, ob $pdx = dy$, erit $2Bydy = A\left(\frac{ydy}{\sqrt{(yy+pp)}} - pd\,\frac{p}{\sqrt{(yy+pp)}}\right)$. At eſt $d.\sqrt{(yy+pp)} = \frac{ydy}{\sqrt{(yy+pp)}} + \frac{pdp}{\sqrt{(yy+pp)}}$ & $\frac{ydy}{\sqrt{(yy+pp)}} = d.\sqrt{(yy+pp)} - \frac{p}{\sqrt{(yy+pp)}}.\, dp$; unde fiet $2Bydy = A\left(d.\sqrt{(yy+pp)} - d.\, \frac{pp}{\sqrt{(yy+pp)}}\right)$, ob $pd\,\frac{p}{\sqrt{(yy+pp)}} + dp\,\frac{p}{\sqrt{(yy+pp)}} = d.\, p.\, \frac{p}{\sqrt{(yy+pp)}} = d.\, \frac{pp}{\sqrt{(yy+pp)}}$. Quare integrando habebitur, ſi $\frac{A}{B} = c$ ponatur, iſta æquatio $yy \pm bb = c\sqrt{(yy+pp)} - \frac{cpp}{\sqrt{(yy+pp)}} = \frac{cyy}{\sqrt{(yy+pp)}}$ ſeu $p = \frac{y\sqrt{(c^2y^2 - (yy\pm bb)^2)}}{yy\pm bb} = \frac{dy}{dx}$; hincque $dx = \frac{(yy\pm bb)dy}{y\sqrt{(c^2y^2 - (yy\pm bb)^2)}}$: ex qua æquatione facile deduci poteſt, ſi ſit $cc \mp 4bb$ quantitas poſitiva, conſtructionem per quadraturam Circuli abſolvi poſſe. At idem facilius patebit, ſi loco dx, vel p, introducamus perpendiculum CP, ex C in tangentem MP demiſſum. Quod ſi autem hoc perpendiculum CP ponatur $= u$, erit $y : u = dx\sqrt{(yy+pp)} : ydx$, hincque $\frac{yy}{\sqrt{(yy+pp)}} = u$; quamobrem

Digitized by Google

brem cum eſſet $yy \pm bb = \frac{cyy}{\sqrt{(yy + pp)}}$ erit $yy \pm bb = cu$, quàm conſtat eſſe aquationem ad ipſum Circulum. Hoc ut oſtendamus, ſumatur Circulus quicunque, centro O, radio $OM = g$ deſcriptus, punctumque C ſumtum ſit in C, ita ut ſit $OC = h$. Jam ducta recta $CM = y$, & $CP = u$, perpendiculari in tangentem MP, erit CP parallela radio OM. Ex M ducatur diametro EF parallela MR, erit $MR = CO = h$; $CR = OM = g$, & $PR = u - g$: quia igitur eſt $MR^2 = MP^2 + PR^2 = CM^2 - CP^2 + PR^2$, erit $h^2 = y^2 - u^2 + (u - g)^2 = y^2 - 2gu + gg$: hincque $yy + gg - hh = 2gu$; quæ comparata cum inventa $yy \pm bb = cu$, fiet $g = \frac{1}{2}c$, & $\pm bb = \frac{1}{4}cc - hh$, ſeu $hh = \frac{1}{4}cc \mp bb$. Erit itaque curva quæſita Circulus, radio $= \frac{1}{2}c$ deſcriptus, puncto C ubi libuerit accepto. In tali Circulo quæſito ſatisfaciet arcus AM, ſi fuerit $\frac{ACM}{AM} = \frac{A}{2b} = \frac{1}{2}c =$ radio OM; hoc eſt ſi fuerit area ACM $=$ arc. AM. AO $=$ duplici ſectori AOM. Hoc autem fieri nequit, niſi punctum C extra Circulum accipiatur; quo caſu hæc conditio infinitis modis adimpleri poteſt; atque adeo effici ut curva ſatisfaciens per data duo puncta tranſeat. Fig. 13.

EXEMPLUM III.

29. *Invenire curvam* DAD *ad axem* AC *relatam, in qua pro data abſciſſa* $AC = a$, *ſit* $\frac{\int x dx \sqrt{(1 + pp)}}{\int dx \sqrt{(1 + pp)}}$ *minimum.* Fig. 14.

Si ponatur abſciſſa indefinita $AP = x$, applicata $PM = y$, & $dy = p\,dx$; exprimit $\frac{\int x dx \sqrt{(1 + pp)}}{\int dx \sqrt{(1 + pp)}}$ diſtantiam centri gravitatis curvæ MAM, tanquam uniformiter gravis ſpectatæ a puncto infimo A; quæ ergo diſtantia, tranſlato P in C, debet eſſe minima. Ad hoc inveniendum, poſito $x = a$, ſit $\int x dx \sqrt{(1 + pp)} = A$, & $\int dx \sqrt{(1 + pp)} = B$: formulæ autem $\int x dx \sqrt{(1 + pp)}$

reperitur valor differentialis $dA = - nv.d.\frac{xp}{\surd(1+pp)}$, & formulæ $\int dx\surd(1+pp)$ valor differentialis $dB = - nv.d.\frac{p}{\surd(1+pp)}$; quibus in æquatione $BdA = AdB$ ſubſtitutis, prodibit $Bd.\frac{xp}{\surd(1+pp)} = Ad.\frac{p}{\surd(1+pp)}$, & poſito $\frac{A}{B} = c$, erit $d.\frac{xp}{\surd(1+pp)} = cd.\frac{p}{\surd(1+pp)}$; unde integrando oritur $\frac{xp}{\surd(1+pp)} = \frac{cp}{\surd(1+pp)} - b$, ſeu $b\surd(1+pp) = (c-x)p$; hincque elicitur $p = \frac{b}{\surd((c-x)^2 - bb)} = \frac{dy}{dx}$. Erit ergo $y = \int\frac{bdx}{\surd((c-x)^2 - b^2)}$; quæ æquatio indicat curvam quæſitam eſſe Catenariam, initio abſciſſarum pro x in loco axis AC quocunque accepto: quin etiam pro axe ſumi poteſt recta quæcunque diametro Catenariæ AC parallela, in eaque punctum quodcunque pro axis initio. Quomodocunque autem axis, ejuſque initium conſtituatur, quæſtioni ſatisfiet ea tantum curvæ portio, ubi fit $\int xdx\surd(1+pp) = c\int dx\surd(1+pp)$. Ponamus pro axe ipſam diametrum AC, & verticem A pro initio abſciſſarum accipi. Quia in A, ubi eſt $x = 0$, fit $\frac{dy}{dx} = p = \infty$, neceſſe eſt ut ſit $cc - bb = 0$, ideoque $b = c$. Verum hoc caſu fit $y = \int\frac{cdx}{\surd(xx - 2cx)}$, quæ curva ſurſum directa fit imaginaria, donec fiat $x > 2c$. Sit ergo $x = 2c + t$, erit $t =$ abſciſſæ AP, & $y = \text{PM} = \int\frac{cdt}{\surd(2ct + tt)}$; curvaque DAD erit catenaria ordinaria. Quo autem appareat quanta ejus portio quæſtioni ſatisfaciat, notandum eſt, ob $dx = dt$, eſſe $p = \frac{c}{\surd(2ct + tt)}$, & $\surd(1+pp) = \frac{c+t}{\surd(2ct+tt)}$; hincque $\int dx\surd(1+pp) = \int\frac{(c+t)dt}{\surd(2ct+tt)} = \surd(2ct+tt)$. At ipſa expreſſio $\frac{\int xdx\surd(1+pp)}{\int dx\surd(1+pp)}$ fit

fit $= 2c + \frac{\int t\,dt\sqrt{(1+pp)}}{\sqrt{(2ct+tt)}}$, quæ ipſi c æqualis fieri nullo modo poteſt. Ex quo concluditur nullam curvæ hujus portionem quæſito præ reliquis magis ſatisfacere. Quamobrem initium abſciſſarum non ſumi poteſt in vertice A. Sumatur ergo in alio quocunque puncto, poſitaque $AP = t$, fieri debet $2bt + tt = (c-x)^2 - bb$; unde fit, vel $b+t = x-c$, vel $b+t = c-x$. Prior æquatio $x = b+c+t$ locum habere nequit; quia, ob $dx = dt$, fieri non poteſt $\frac{\int x\,dt\sqrt{(1+pp)}}{\int dt\sqrt{(1+pp)}}$ ſeu $(b+c+\frac{\int t\,dt\sqrt{(1+pp)}}{\int dt\sqrt{(1+pp)}}) = c$. Ergo fiat $x = c-b-t$, quo caſu abſciſſæ ab aliquo puncto axis AC ſuperiori deorſum deſcendent: fierique deberet $\frac{\int x\,dx\sqrt{(1+pp)}}{\int dx\sqrt{(1+pp)}}$ ſeu $c - b - \frac{\int t\,dt\sqrt{(1+pp)}}{\int dt\sqrt{(1+pp)}} = c$, quod pariter fieri nequit; ex quo concludendum eſt, nullam portionem magis quam aliam quamvis ſatisfacere. Hoc autem inde venire videtur, quod Catenaria duas habet partes conjugatas veluti Hyperbola conica, hincque ſemper fieri poteſt $\frac{\int x\,dx\sqrt{(1+pp)}}{\int dx\sqrt{(1+pp)}} = 0$, qui eſt valor minimus. Hoc clarius confirmari poteſt ex valore invento $p = \frac{b}{\sqrt{((c-x)^2 - b^2)}}$; unde fit $\sqrt{(1+pp)} = \frac{c-x}{\sqrt{((c-x)^2-b^2)}} = (c-x)r$, poſito brevitatis gratia $r = \frac{1}{\sqrt{((c-x)^2-b^2)}}$. Oporteret ergo in caſu quæſito eſſe $\frac{\int(c-x)xr\,dx}{\int(c-x)r\,dx} = c$ ſeu $\int(c-x)^2 r\,dx = 0$, quod cum caſu $x = 0$ evaneſcere debeat, alio inſuper caſu evaneſcere deberet. At eſt $\int(c-x)^2 r\,dx = \int\frac{(c-x)^2\,dx}{\sqrt{((c-x)^2-b^2)}}$ $= -\frac{1}{2}(c-x)\sqrt{((c-x)^2-b^2)} - \frac{bb}{2}l\frac{c-x+\sqrt{((c-x)^2-b^2)}}{c+\sqrt{(c^2-b^2)}}$ $+ \frac{1}{2}c\sqrt{(c^2-b^2)}$, quæ expreſſio, cum ſemel fuit $= 0$, poſt, ob $(c-x)^2$ perpetuo affirmativum, continuo creſcet neque denuo

nuo fieri poteſt $= 0$. Quamobrem ambos terminos integrationis formulæ $\int \frac{(c-x)^2 dx}{\sqrt{((c-x)^2 - b^2)}}$ inter ſe congruere oportet; quod evenit ſi fuerit $x = c$: quo caſu curva ſatisfaciens abit in lineam rectam axi normalem, quæ utique centrum ſuum gravitatis a ſe minime habet remotum.

Exemplum IV.

30. *Invenire curvam, in qua, pro data abſciſſa* $x = a$, *ſit hæc expreſſio* $\frac{\int y x dx}{\int dx \sqrt{(1+pp)}}$ *maximum vel minimum.*

Poſito $x = a$, fiet $\int y x dx = A$, & $\int dx \sqrt{(1+pp)} = B$. Jam formulæ $\int y x dx$ valor differentialis eſt $dA = nv.dx.x = nv.xdx$, & formulæ $\int dx \sqrt{(1+pp)}$ valor differentialis eſt $dB = -nv.d.\frac{p}{\sqrt{(1+pp)}}$. Quare cum ſit $BdA = AdB$, habebitur iſta æquatio $Bxdx = -d.\frac{p}{\sqrt{(1+pp)}}$, ſeu $xdx = -ccd.\frac{p}{\sqrt{(1+pp)}}$, poſito $A = Bc^2$. Unde integrando obtinebitur $xx = C - \frac{2ccp}{\sqrt{(1+pp)}}$, hincque $p = \frac{bc - xx}{\sqrt{(4c^4 - (bc-xx)^2)}} = \frac{dy}{dx}$; quæ præbet $y = \int \frac{(bc-xx)dx}{\sqrt{(4c^4-(bc-xx)^2)}}$, quæ eſt æquatio generalis pro curva Elaſtica: cujus hæc proprietas, quod radius oſculi ubique abſciſſæ x ſit reciproce proportionalis: id quod patet ex æquatione $xdx = -ccd.\frac{p}{\sqrt{(1+pp)}}$, quæ abit in $\frac{-dx}{d.\frac{p}{\sqrt{(1+pp)}}} = \frac{cc}{x}$, eſtque $\frac{-dx}{d.\frac{p}{\sqrt{(1+pp)}}} = -\frac{dx(1+pp)^{3:2}}{dp}$ radius oſculi in curva. Hujus autem curvæ tanta portio ab initio computando ſatisfacit, in qua erit $\int yxdx =$

$= ccſdx\sqrt{(1+pp)} = 2c^4ſ\frac{dx}{\sqrt{(4c^4-(bc-xx)^2)}}$; quæ determinatio eo revocatur, ut effici debeat $ſdx\sqrt{(4c^4-(bc-xx)^2)} = (aa-bc)ſ\frac{dx(bc-xx)}{\sqrt{(4c^4-(bc-xx)^2)}}$, ſi poſt integrationem utramque ponatur $x=a$. Hoc itaque modo conſtans illa c per a determinabitur.

Propositio IV. Problema.

31. *Invenire æquationem inter binas variabiles* x & y *ita comparatam, ut, poſita variabili* x = a, *maximum minimumve fiat expreſſio* W, *quæ ſit functio quæcunque formularum integralium* ſZdx, ſYdx, ſXdx, *&c. in quibus denotent* Z, Y, X *&c. functiones quaſcunque ipſarum* x, y, p, q, *&c. ſive, determinatas, ſive indeterminatas.*

Solutio.

Ponamus idoneam æquationem inter x & y jam eſſe inventam, poſitoque $x=a$, fieri $ſZdx=A$; $ſYdx=B$; $ſXdx=C$ &c. hisque valoribus in expreſſione W ſubſtitutis, habebitur revera maximum vel minimum. Quod ſi igitur altera variabilis y in uno loco particula $n\nu$ augeri ponatur, atque naſcentes hinc mutationes in ſingulis formulis $ſZdx$, $ſYdx$, $ſXdx$ &c. introducantur, idem pro W valor prodire debet. At ab illa particula $n\nu$ formulæ $ſZdx$, $ſYdx$, & $ſXdx$ &c. quæque ſuis valoribus differentialibus augebuntur. Si ergo ponatur formulæ $ſZdx$ valor differentialis $=dA$, formulæ $ſYdx=dB$, formulæ $ſXdx=dC$, &c. loco quantitatum A, B, C, &c. orientur a particula $n\nu$ iſtæ auctæ $A+dA$, $B+dB$, $C+dC$ &c. quæ in W ſubſtitutæ eundem valorem producere debent, quem ipſæ A, B, C, &c. Ponamus, $A+dA$, $B+dB$, $C+dC$ &c. loco $ſZdx$, $ſYdx$, $ſXdx$ &c. ſubſtitutis, prodire $W+dW$; eritque $W+dW=W$, ideoque $dW=0$. Hic autem valor dW, ut ex differentiationis natura liquet, invenitur,

tur, ſi quantitas W, poſtquam in illa, loco formularum integralium, litteræ A, B, C &c. ſunt ſubſtitutæ, differentietur, his ipſis litteris A, B, C, &c. tanquam variabilibus tractatis; in hocque differentiali, dA, dB, dC &c. valores differentiales formularum reſpondentium $\int Zdx$, $\int Tdx$, $\int Xdx$ &c. deſignent. Hac igitur ſignificatione ſumptum differentiale quantitatis propoſitæ W, ſi id nihilo æquale ponatur, dabit æquationem inter x & y quæſitam. Q. E. I.

Coroll. I.

32. Si ergo propoſita fuerit ejuſmodi expreſſio W functio formularum integralium $\int Zdx$, $\int Tdx$, $\int Xdx$ &c. quæ, pro determinato ipſius x valore $= a$, debeat eſſe maximum vel minimum: tum loco formularum $\int Zdx$, $\int Tdx$, $\int Xdx$ &c. ſcribantur litteræ A, B, C, &c. quo facto, expreſſio W differentietur his litteris A, B, C, &c. ſolis tanquam variabilibus tractatis, atque differentiale ponatur $= 0$.

Coroll. II.

33. In hoc differentiali, in quo inerunt litteræ A, B, C, &c. cum ſuis differentialibus dA, dB, dC &c. litteræ A, B, C &c. denotabunt reſpective valores formularum $\int Zdx$, $\int Tdx$, $\int Xdx$ &c. quos induunt poſito $x = a$; at differentialia dA, dB, dC, &c. exprimunt valores differentiales earundem formularum integralium abſciſſæ $x = a$ reſpondentes.

Coroll. III.

34. Ex præcedentibus autem apparet, ſi Z, T, X &c. fuerint functiones determinatæ quantitatum x, y, p, q, &c. tum valores differentiales dA, dB, dC, &c. non a valore a pendere: contra vero ſi Z, T, X &c. fuerint functiones indefinitæ, tum valores differentiales dA, dB, dC &c. ſimul a valore a pendere debere.

Digitized by Google

COROLL. IV.

35. Cum igitur hoc modo W fiat functio litterarum A, B, C, &c. ejus differentiale hujuſmodi habebit formam $FdA + GdB + HdC +$ &c. hincque æquatio quæſita erit $o = FdA + GdB + HdC +$ &c. ubi F, G, H &c. erunt quantitates conſtantes, per A, B, C &c. determinatæ.

COROLL. V.

36. Æquatio ergo Problemati ſatisfaciens, conſtabit ex valoribus differentialibus ſingularum formularum integralium in maximi minimive expreſſione W contentarum, ſingulis per conſtantes quantitates determinatas multiplicatis: horum ſcilicet productorum aggregatum nihilo æquale poſitum dabit æquationem deſideratam.

SCHOLION I.

37. Potuiſſemus hanc Problema propoſitum reſolvendi methodum, ex ſolutionibus binorum Problematum præcedentium, per inductionem, jam concludere: quippe ex quibus jam patebat, ſi fuerit maximi minimive formula W, vel productum ex duabus formulis integralibus, vel quotus ex diviſione unius per alteram ortus, tum differentiale expreſſionis W modo expoſito ſumtum præbere æquationem Problemati convenientem. Præſtitit autem hoc Problema, ob ſummam ejus extenſionem, ſingulari ſolutione munire. In hoc enim Problemate continentur omnes omnino Quæſtiones, quæ in hoc genere, quo expreſſio quæpiam maxima minimave deſideratur, unquam proponi atque excogitari poſſunt: ideoque per iſtam Propoſitionem penitus exhauſta eſt methodus maximorum ac minimorum abſoluta, quam primo pertractandam ſuſcepimus. Præterea hîc notandum eſt, ſi expreſſio W non tantum formulas integrales, uti poſui-

 mus,

mus, complectatur; verum etiam functiones determinatas ipſarum x, y, p, q, &c. tum ſolutionem nihilo difficiliorem reddi. Nam pari modo, loco harum functionum determinatarum, quantitates conſtantes poni debent, in quas ſcilicet abeunt poſito $x = a$; at poſtmodum, in differentiatione ipſius W, has quantitates etiam tanquam conſtantes tractari oportet; eo quod functiones determinatæ nullos valores differentiales recipiunt. Quo autem clarius appareat, quomodo iſtiuſmodi expreſſiones tractari conveniat; in ſequentibus Exemplis nonnulla occurrent, quæ hoc argumentum penitus illuſtrabunt.

EXEMPLUM I.

38. *Invenire curvam coordinatis orthogonalibus contentam, in qua ſit maximum vel minimum iſta expreſſio* $(1+pp)^{1:2}\int y\,dx + y\int dx\sqrt{(1+pp)}$; *ſi ponatur abſciſſa* $x = a$.

Ponamus æquationem inter x & y quæſito ſatisfacientem jam eſſe inventam, atque poſito $x = a$ fieri $y = f$ & $\sqrt{(1+pp)} = g$: itemque $\int y\,dx = A$, & $\int dx\sqrt{(1+pp)} = B$; erit $dA = nv.dx$, & $dB = -nv.d\frac{p}{\sqrt{(1+pp)}}$. Expreſſio igitur, quæ maxima erit vel minima, hoc caſu eſt $gA + fB$, cujus differentiale eſt $g\,dA + f\,dB$; quod poſitum $= 0$, dabit æquationem deſideratam pro curva. Hic ſcilicet intelligitur litteras y & f, quæ ex functionibus determinatis ſunt ortæ, in differentiatione tanquam quantitates conſtantes eſſe tractatas. Subſtitutis jam pro dA & dB valoribus debitis, diviſioneque per nv facta, orietur iſta æquatio pro curva quæſita $g\,dx = f\,d.\frac{p}{\sqrt{(1+pp)}}$. Ponatur $\frac{f}{g} = c$, ita ut ſit $\frac{y}{\sqrt{(1+pp)}} = c$, caſu quo eſt $x = a$; erit integrando $x + b = \frac{cp}{\sqrt{(1+pp)}}$, atque $p =$

$\frac{x+b}{\sqrt{(cc-(x+b)^2)}} = \frac{dy}{dx}$; ex qua fit $y = h \pm \sqrt{(c^2 - (x+b)^2)}$. Curva igitur ſatisfaciens eſt Circulus, radio c deſcriptus, abſciſſis ſuper recta quacunque aſſumptis, pariterque abſciſſarum initio ubicumque ſtatuto. Quantitas autem c, quæ radium Circuli conſtituit, ex definita abſciſſa $x = a$ determinatur; quia eſſe debet $\frac{y}{\sqrt{(1+pp)}} = c$, caſu quo $x = a$. Fit autem hoc caſu $y = h \pm \sqrt{(c^2 - (a+b)^2)}$, & $\sqrt{(1+pp)} = \frac{c}{\sqrt{(cc-(a+b)^2)}}$: unde oritur $cc = h\sqrt{(cc-(a+b)^2)} \pm (cc - (a+b)^2)$, per quam, vel c per a, vel viciſſim a per c determinari poteſt. Ponamus eſſe $h = 0$, $b = -c$, ita ut axis ſit Circuli diameter, initiumque abſciſſarum in vertice conſtituatur; erit $y = \sqrt{(2cx - xx)}$, atque fiet $(a-c)^2 = 0$, ſeu $c = a$. Ex quo intelligitur, hoc caſu quadrantem Circuli quæſito ſatisfacere. Sin autem initium abſciſſarum in loco diametri quocunque capiatur, fiet tantum $h = 0$, & ſi applicatæ poſitivæ ſumantur fiet $(a+b)^2 = 0$, ſeu, $b = -a$. Diameter Circuli ergo manet indeterminatus: portioque Circuli hoc modo ſumti quæſtioni ſatisfaciet, quæ abſciſſæ a ſua origine ad centrum Circuli uſque productæ reſpondet.

EXEMPLUM II.

39. *Invenire æquationem inter* x *&* y, *ut pro valore definito* x $= a$, *hæc expreſſio* $y^{\int dx \sqrt{(1+pp)}} \int y dx$ *fiat maximum vel minimum.*

Poſito $x = a$, fiat $y = f$, $\int dx \sqrt{(1+pp)} = A$, & $\int y dx = B$, erit $dA = -nv.\ d.\ \frac{p}{\sqrt{(1+pp)}}$ & $dB = nv.\ dx$. Maximum ergo minimumve eſſe oportet hanc quantitatem $f^A B$, cujus differentiale eſt $f^A B dA lf + f^A dB$; quod poſitum $= 0$ dabit $B dA lf = - dB$. Pro æquatione quæſita igitur habetur

Digitized by Google

betur $Blfd. \frac{p}{\sqrt{(1+pp)}} = dx$, & integrando $x+b = \frac{Bplf}{\sqrt{(1+pp)}}$ $= \frac{cp}{\sqrt{(1+pp)}}$, poſito $Blf = c$. Habetur ergo $p = \frac{b+x}{\sqrt{(cc-(b+x)^2)}}$ & $y = h \pm \sqrt{(c^2-(b+x)^2)}$. Erit igitur $f = h \pm \sqrt{(c^2-(b+a)^2)}$, poſito $x = a$, atque $B = \int y\,dx = ha \pm \int dx \sqrt{(c^2-(b+x)^2)}$, poſito poſt integrationem $x = a$. Facto igitur $Blf = c$, innoteſcet valor a, cui ſi x æqualis capiatur in Circulo radii c, portio abſcindetur Problemati ſatisfaciens. Cæterum ex his & Coroll. 5 colligere licet, quoties formula maximi minimive fuerit functio quæcunque binarum harum formularum $\int y\,dx$ & $\int dx\sqrt{(1+pp)}$, curvam ſatisfacientem perpetuo eſſe Circulum: tantum ex ſolutione quantitas portionis ſatisfacientis debet diligenter inveſtigari ac determinari.

Exemplum III.

40. *Invenire æquationem inter* x *&* y, *ut, poſito* x = a, *maximum minimumve fiat iſta expreſſio* $e^{-n\int dx\sqrt{(1+pp)}} \int e^{n\int dx\sqrt{(1+pp)}}\, dx$.

Ponamus, caſu propoſito quo $x = a$, fieri $n\int dx\sqrt{(1+pp)} = A$, atque $\int e^{n\int dx\sqrt{(1+pp)}}\,dx = B$; ita ut maximum minimumve ſit hæc quantitas $e^{-A}B$, cujus differentiale eſt $e^{-A}dB - e^{-A}BdA$; quod poſitum $= 0$ dabit æquationem hanc $dB = BdA$. At eſt dA valor differentialis formulæ $n\int dx\sqrt{(1+pp)}$, unde erit $dA = -\,n\,\nu.\,d.\,\frac{np}{\sqrt{(1+pp)}}$: atque dB eſt valor differentialis formulæ $\int e^{n\int dx\sqrt{(1+pp)}}dx$; quæ continetur in Caſu ſecundo §. 7, ubi eſt $Z = e^{n\int dx\sqrt{(1+pp)}}$ & $\Pi = \int dx\sqrt{(1+pp)}$, ita ut ſit $Z = e^{n\Pi}$, & $dZ = e^{n\Pi}n\,d\Pi$, unde erit $L = e^{n\Pi}n$, & reliquæ litteræ M, N, P,

Digitized by Google

P, &c. fient $= 0$. Porro, ob $\Pi = \int dx \surd(1+pp)$, erit $[Z] = \surd(1+pp)$, & $d[Z] = \frac{p\,dp}{\surd(1+pp)}$, ex quo erit $[M] = 0$, $[N] = 0$, & $[P] = \frac{p}{\surd(1+pp)}$. Jam eſt $\int L\,dx = n\int e^{n\int dx \surd(1+pp)}\,dx$, cujus valor, poſito $x = a$, erit $= nB$; hincque $V = n\,(B - \int e^{n\int dx \surd(1+pp)}\,dx)$. Per Regulam ergo datam fiet $dB = n\nu.\ dx(-\frac{d.\,[P]\,V}{dx}) = -n\nu.\ d.\,\frac{np(B - \int e^{n\int dx \surd(1+pp)}\,dx)}{\surd(1+pp)} = -n\nu.\ d.\,\frac{nBp}{\surd(1+pp)}$, ob $dB = B\,dA$. Integrando itaque erit $\frac{np(B - \int e^{n\int dx \surd(1+pp)}\,dx)}{\surd(1+pp)} = \frac{nBp}{\surd(1+pp)} - nb$; hincque $\frac{b\surd(1+pp)}{p} = \int e^{n\int dx \surd(1+pp)}\,dx$. Ex qua æquatione, quia valor determinatus a exceſſit, perſpicuum eſt æquationem inventam pro quovis ipſius x valore æque valere. Ut autem hanc æquationem evolvamus, erit, differentialibus ſumtis, $\frac{-b\,dp}{p^2\surd(1+pp)} = e^{n\int dx \surd(1+pp)}\,dx$: quæ, per $\surd(1+pp)$ multiplicata atque integrata, dat $\frac{nb}{p} + c = e^{n\int dx \surd(1+pp)}$, qui exponentialis quantitatis valor in illa æquatione ſubſtitutus, dabit $\frac{nb\,dx}{p} + c\,dx = -\frac{b\,dp}{p^2\surd(1+pp)}$, ſeu $dx = \frac{-b\,dp}{p(nb+cp)\surd(1+pp)}$. Commodior autem æquatio oritur, ſi ponatur $\int dx\surd(1+pp) = s$, eritque s arcus curvæ, ſi fuerint x & y coordinatæ normales. Quare habebitur iſta æquatio $nb + cp = e^{ns}p$, quæ per dx multiplicata, ob $dy = p\,dx$, abit in hanc $nb\,dx + c\,dy = e^{ns}\,dy$.

Cum

Digitized by Google

Cum autem, poſito, $x = 0$, arcus s evaneſcere debeat, neceſſe eſt ut ſit hoc caſu $\frac{nb}{p} + c = 0$; hinc itaque vel dato curvæ initio conſtans c determinabitur, vel viciſſim ex c poſitio primæ tangentis innoteſcet. Cæterum ſi hanc quæſtionem attentius contemplemur, deprehendemus eam jam contineri in Exemplo quodam Capitis præced. §. 45. Cum enim noſtra expreſſio, quæ maximum minimumve eſſe debeat, ſit $e^{-n\int dx\sqrt{(1+pp)}} \int e^{n\int dx\sqrt{(1+pp)}} dx$; ponatur ea $= W$, erit $e^{n\int dx\sqrt{(1+pp)}} W = \int e^{n\int dx\sqrt{(1+pp)}} dx$, atque differentiando fiet $dW + nWdx\sqrt{(1+pp)} = dx$. Maximi igitur minimive expreſſio W datur per æquationem differentialem, quæ in Caſu quarto §. 7 continetur: atque methodo convenienti tractata ad eandem perducit æquationem, quam hic invenimus. Quæſtionem autem illam in ſe complectentem ſupra in Cap. præc. §. 45 tractavimus, in quo hunc ipſum caſum adjunctum ſpectare licet. Comparatione autem inſtituta, ſummus perſpicietur conſenſus ſolutionum variarum ejuſdem Problematis, quæ quidem tentari queant.

EXEMPLUM IV.

41. *Invenire curvam in qua, pro data abciſſa* $= a$, *fiat iſta expreſſio* $\frac{\int dx\, \text{ſin.}\, A.\, y.\, \sqrt{(1+pp)}}{\int dx\, \text{coſ.}\, A\, y.\, \sqrt{(1+pp)}}$ *maximum vel minimum.*

Poſito $x = a$, fiat $\int dx(1+pp)^{1:2}\, \text{ſin.}\, Ay = A$, & $\int dx(1+pp)^{\frac{1}{2}}\, \text{coſ.}\, Ay = B$; erit, per valores differentiales, $dA = nv.\, dx((1+pp)^{1:2}\, \text{coſ.}\, Ay - \frac{1}{dx}\, d.\, \frac{p\, \text{ſin.}\, A.\, y}{\sqrt{(1+pp)}})$, & $dB = nv.\, dx(-(1+pp)^{1:2}\, \text{ſin.}\, Ay - \frac{1}{dx}\, d.\, \frac{p\, \text{coſ.}\, Ay}{\sqrt{(1+pp)}})$. Cum igitur $\frac{A}{B}$ debeat eſſe maximum vel minimum, erit BdA

=

$= A\,dB$; poſito ergo $\frac{A}{B} = m$, fiet $(1+pp)^{1:2}\,dx\,\text{coſ.}\,Ay - d.\,\frac{p\,\text{ſin.}\,Ay}{\sqrt{(1+pp)}} = -m(1+pp)^{\frac{1}{2}}\,dx\,\text{ſin.}\,Ay - md.\,\frac{p\,\text{coſ.}\,Ay}{\sqrt{(1+pp)}}$. Multiplicetur per p, erit [ob $d.\,(1+pp)^{1:2}\,\text{ſin.}\,Ay = dy(1+pp)^{1:2}\,\text{coſ.}\,Ay + \frac{p\,dp\,\text{ſin}\,Ay}{\sqrt{(1+pp)}}$, & $d.\,(1+pp)^{1:2}\,\text{coſ.}\,Ay = -dy(1+pp)^{1:2}\,\text{ſin}\,A.y + \frac{p\,dp\,\text{coſ.}\,Ay}{\sqrt{(1+pp)}}$] $d.\,(1+pp)^{1:2}\,\text{ſin.}\,Ay - d.\,\frac{pp\,\text{ſin.}\,Ay}{\sqrt{(1+pp)}} = md.\,(1+pp)^{1:2}\,\text{coſ.}\,Ay - md.\,\frac{pp\,\text{coſ.}\,Ay}{\sqrt{(1+pp)}}$; quæ integrata & reducta præbet $\frac{\text{ſin.}\,Ay}{\sqrt{(1+pp)}} = \frac{m\,\text{coſ.}\,Ay}{\sqrt{(1+pp)}} + b$, ſive $b\sqrt{(1+pp)} = \text{ſin.}\,Ay - m\,\text{coſ.}\,Ay$; ubi notandum eſt fieri debere, ſi $x = a$ ponatur, $m = \frac{\int dx(1+pp)^{1:2}\,\text{ſin.}\,Ay}{\int dx(1+pp)^{1:2}\,\text{coſ.}\,Ay}$. Sit $m = \frac{\text{ſin.}\,An}{\text{coſ.}\,An} = \text{tang.}\,An$; fiet $b\sqrt{(1+pp)} = \frac{\text{ſin}\,A(y-n)}{\text{coſ.}\,An}$, atque $y = n + A\,\text{ſin.}\,b(1+pp)^{1:2}\,\text{coſ.}\,An$. Quia vero eſt $dy = pdx$, erit $dx = \frac{dy}{p}$. At eſt $dy = \frac{cp\,dp}{\sqrt{(1+pp)(1-cc-ccpp)}}$, poſito $b\,\text{coſ.}\,An = c$. Ex quibus conficitur $x = \int\frac{c\,dp}{\sqrt{(1+pp)(1-cc-ccpp)}}$, atque $y = \int\frac{cp\,dp}{\sqrt{(1+pp)(1-cc-ccpp)}}$; longitudo autem curvæ erit $= \int\frac{c\,dp}{\sqrt{(1-cc-ccpp)}} = A\,\text{ſin.}\,\frac{cp}{\sqrt{(1-cc)}}$. Quare ſi arcus curvæ dicatur s; habebitur iſta concinna æquatio $dx\,\text{ſin.}\,As = \frac{c\,dy}{\sqrt{(1-cc)}}$: Conſtructio vero ex anterioribus formulis ſponte conſequitur.

Cum autem, poſito, $x = 0$, arcus s evaneſcere debeat, neceſſe eſt ut ſit hoc caſu $\frac{nb}{p} + c = 0$; hinc itaque vel dato curvæ initio conſtans c determinabitur, vel viciſſim ex c poſitio primæ tangentis innoteſcet. Cæterum ſi hanc quæſtionem attentius contemplemur, deprehendemus eam jam contineri in Exemplo quodam Capitis præced. §. 45. Cum enim noſtra expreſſio, quæ maximum minimumve eſſe debeat, ſit $e^{-n\int dx\surd(1+pp)} \int e^{n\int dx\surd(1+pp)}\,dx$; ponatur ea $= W$, erit $e^{n\int dx\surd(1+pp)}W = \int e^{n\int dx\surd(1+pp)}\,dx$, atque differentiando fiet $dW + nW\,dx\surd(1+pp) = dx$. Maximi igitur minimive expreſſio W datur per æquationem differentialem, quæ in Caſu quarto §. 7 continetur: atque methodo convenienti tractata ad eandem perducit æquationem, quam hîc invenimus. Quæſtionem autem illam in ſe complectentem ſupra in Cap. præc. §. 45 tractavimus, in quo hunc ipſum caſum adjunctum ſpectare licet. Comparatione autem inſtituta, ſummus perſpicietur conſenſus ſolutionum variarum ejuſdem Problematis, quæ quidem tentari queant.

EXEMPLUM IV.

41. *Invenire curvam in qua, pro data abciſſa* $= a$, *fiat iſta expreſſio* $\frac{\int dx\,\text{ſin.}\,A.\,y.\,\surd(1+pp)}{\int dx\,\text{coſ.}\,A\,y.\,\surd(1+pp)}$ *maximum vel minimum.*

Poſito $x = a$, fiat $\int dx(1+pp)^{1:2}\,\text{ſin.}\,A\,y = A$, & $\int dx(1+pp)^{\frac{1}{2}}\,\text{coſ.}\,A\,y = B$; erit, per valores differentiales, $dA = nv.\,dx((1+pp)^{1:2}\,\text{coſ.}\,Ay - \frac{1}{dx}\,d.\,\frac{p\,\text{ſin.}\,A.\,y}{\surd(1+pp)})$, & $dB = nv.\,dx(-(1+pp)^{1:2}\,\text{ſin.}\,Ay - \frac{1}{dx}\,d.\,\frac{p\,\text{coſ.}\,Ay}{\surd(1+pp)})$. Cum igitur $\frac{A}{B}$ debeat eſſe maximum vel minimum, erit $B\,dA$

=

$= A\,dB$; poſito ergo $\frac{A}{B} = m$, fiet $(1+pp)^{1:2}dx \cos. Ay - d.\frac{p \sin. Ay}{\sqrt{(1+pp)}} = -m(1+pp)^{\frac{1}{2}}dx \sin. Ay - md.\frac{p \cos. Ay}{\sqrt{(1+pp)}}$. Multiplicetur per p, erit [ob $d.(1+pp)^{1:2}\sin. Ay = dy(1+pp)^{1:2}\cos. Ay + \frac{pdp \sin Ay}{\sqrt{(1+pp)}}$, & $d.(1+pp)^{1:2}\cos. Ay = -dy(1+pp)^{1:2}\sin A.y + \frac{pdp \cos. Ay}{\sqrt{(1+pp)}}$] $d.(1+pp)^{1:2}\sin. Ay - d.\frac{pp \sin. Ay}{\sqrt{(1+pp)}} = md.(1+pp)^{1:2}\cos. Ay - md.\frac{pp \cos. Ay}{\sqrt{(1+pp)}}$; quæ integrata & reducta præbet $\frac{\sin. Ay}{\sqrt{(1+pp)}} = \frac{m \cos. Ay}{\sqrt{(1+pp)}} + b$, ſive $b\sqrt{(1+pp)} = \sin. Ay - m \cos. Ay$; ubi notandum eſt fieri debere, ſi $x = a$ ponatur, $m = \frac{\int dx(1+pp)^{1:2}\sin. Ay}{\int dx(1+pp)^{1:2}\cos. Ay}$. Sit $m = \frac{\sin. An}{\cos. An} = \text{tang. } An$; fiet $b\sqrt{(1+pp)} = \frac{\sin A(y-n)}{\cos. An}$, atque $y = n + A \sin. b(1+pp)^{1:2}\cos. An$. Quia vero eſt $dy = pdx$, erit $dx = \frac{dy}{p}$. At eſt $dy = \frac{cpdp}{\sqrt{(1+pp)(1-cc-ccpp)}}$, poſito $b \cos. An = c$. Ex quibus conficitur $x = \int \frac{cdp}{\sqrt{(1+pp)(1-cc-ccpp)}}$, atque $y = \int \frac{cpdp}{\sqrt{(1+pp)(1-cc-ccpp)}}$; longitudo autem curvæ erit $= \int \frac{cdp}{\sqrt{1-cc-ccpp}} = A \sin. \frac{cp}{\sqrt{(1-cc)}}$. Quare ſi arcus curvæ dicatur s; habebitur iſta concinna æquatio $dx \sin. As = \frac{cdy}{\sqrt{(1-cc)}}$: Conſtructio vero ex anterioribus formulis ſponte conſequitur.

SCHOLION II.

42. His igitur Capitibus penitus abſolvimus eam Methodi maximorum ac minimorum ad lineas curvas inveniendas accommodatæ partem, quam abſolutam vocavimus: in qua ſemper linea curva requiri ſolet, quæ habeat, pro dato quodam abſciſſæ ſeu alterius variabilis x valore, expreſſionem quamcunque indeterminatam, maximum minimumve. Nam iſta expreſſio, quæ maximum minimumve eſſe debet, vel erit una quædam formula integralis formæ $\int Z\,dx$, ita ut Z ſit functio quæcunque ipſarum x, y, p, q, &c. ſive definita ſive indefinita; pro quibus caſibus Methodum tradidimus in Capitibus præcedentibus: Vel maximi minimive expreſſio illa continebit in ſe plures ejuſmodi formulas integrales, ita ut ſit duarum pluriumve formularum integralium functio quæcunque; pro hocque caſu Methodus idonea in iſto Capite eſt expoſita, atque Exemplis illuſtrata. Univerſa autem Methodus, quam hic dedimus, nititur inventione valorum differentialium, qui ſingulis formulis integralibus quæ vel ipſæ maximum minimumve eſſe debeant, vel in maximi minimive expreſſione contineantur, atque ideo tota ſolvendi Methodus reducitur ad Caſus illos, quos §. 7 hujus Capitis conjunctim repræſentavimus. Qui igitur illos caſus in memoria tenet, vel in promtu habet, is ad omnia hujus generis Problemata expedite reſolvenda erit paratus. Neque vero ſolum Caſus ibi enumerati Methodum maximorum ac minimorum abſolutam conſtituunt; verum etiam Methodum alteram relativam, quam in ſequentibus aggrediemur, abſolvent; ex quo illorum Caſuum ſummus uſus in utraque Methodo abunde perſpicietur. Hanc autem tractationem duobus Capitibus abſolvemus, in quorum priori omnibus curvis, ex quibus quæſita debet erui, unam quandam proprietatem communem, in poſteriori vero plures tribuemus.

CA-

Digitized by Google

CAPUT V.

Methodus, inter omnes curvas eadem proprietate præditas, inveniendi eam, quæ maximi minimive proprietate gaudeat.

DEFINITIO.

1. PRopriètas communis eſt Formula integralis, ſeu expreſſio indefinita, quæ in omnes curvas ex quibus quæſitam determinari oportet, æqualiter competit.

SCHOLION I.

2. Hactenus Methodum maximorum ac minimorum tradidimus abſolutam, in qua perpetuo, inter omnes omnino curvas eidem abſciſſæ reſpondentes, una requiri ſolebat, quæ maximi minimive cujuſpiam proprietate gauderet. Nunc autem progredimur ad Methodum relativam, in qua unam lineam maximi minimive proprietate præditam determinare docebimus, non ex omnibus omnino lineis eidem abſciſſæ reſpondentibus, verum ex illis, innumerabilibus quidem, lineis curvis tantum, quibus una quædam proprietas propoſita pluresve ſint communes. Ac primo quidem, in hoc Capite, innumerabiles curvas eidem abſciſſæ reſpondentes contemplabimur, quæ unam quandam proprietatem habeant communem; ex hisque unam lineam inveſtigabimus, in qua expreſſio quæcunque indefinita maximum minimumve obtineat valorem. Hoc in genere inprimis celebre eſt *Problema Iſoperimetricum*, initio hujus ſæculi publice propoſitum, in quo, inter omnes curvas ejuſdem longitudinis quæ quidem eidem abſciſſæ reſpondeant, eam definiri oportebat, quæ contineret maximi minimive cujuſpiam proprietatem. Poſtmodum autem hæc Quæſtio in latiori ſenſu eſt accepta, ut iſta determina-

Digitized by Google

minatio non ſolum inter omnes curvas ejuſdem longitudinis fieret, verum etiam inter omnes curvas alia quacunque proprietate communi præditas; quam ipſam quæſtionem in hoc Capite pertractare ſuſcepimus. Cum igitur curva ſit eligenda, non ex omnibus omnino curvis eidem abſciſſæ reſpondentibus, verum ex iis, innumerabilibus duntaxat, in quas proprietas quæpiam propoſita æqualiter competat; hanc ipſam proprietatem ante omnia conſiderari oportet, quam hîc nomine proprietatis communis indicamus. Hæc igitur proprietas communis, veluti æqualitas longitudinis curvarum, omnia puncta media afficere debet, & hanc ob rem erit functio indefinita, quæ, non ex unici curvæ elementi, verum ex totius curvæ poſitione determinetur. Quam ob rem iſtiuſmodi proprietas communis erit, vel formula integralis indefinita ſimplex, vel expreſſio plures ejuſmodi formulas integrales complectens. Omnino igitur pari modo erit comparata, quo ipſa maximi minimive formula, ſeu expreſſio. Eædem igitur varietates atque diviſiones, quas ante circa maximi minimive expreſſionem fecimus & tractavimus, æque ad proprietatem communem pertinebunt.

Coroll. I.

3. Si igitur proprietas communis fuerit propoſita, quæ ſit B, tum omnes curvæ ſunt conſiderandæ, quæ pro eadem data abſciſſa eundem valorem ipſius B continent; atque ex his ea debet definiri, quæ habeat maximum vel minimum.

Coroll. II.

4. In Problematis ergo huc pertinentibus duas res datas eſſe oportet, proprietatem communem B, ac maximi minimive expreſſionem A. Quibus datis, inter omnes curvas pro data abſciſſa eundem valorem B continentes, ea definiri debebit, quæ pro eadem abſciſſa valorem ipſius A habeat maximum vel minimum.

Co-

Digitized by Google

COROLL. III.

5. Dantur autem non ſolum infinitæ curvæ, quæ pro data abſciſſa eandem proprietatem communem habeant, ſed etiam dantur infinitis modis. Aſſumta enim curva quacunque pro lubitu, ea determinatum habebit valorem proprietatis communis propoſitæ; præter eam autem dabuntur innumerabiles aliæ eundem valorem proprietatis communis pro eadem abſciſſa continentes.

COROLL. IV.

6. Propoſita igitur expreſſione quacunque indefinita, innumerabilia infinitarum curvarum dabuntur genera; quorum quodlibet genus infinitas in ſe complectitur curvas, quæ pro eadem data abſciſſa eundem illius expreſſionis valorem contineant.

COROLL. V.

7. Cum igitur infinita dentur genera, quorum ſingula innumerabiles lineas curvas comprehendunt, in quas propoſita pro proprietate communi expreſſio æqualiter competat; in uno quoque genere dabitur una curva, quæ, pro reliquis ejuſdem generis curvis, alteram expreſſionem in maximo minimove gradu contineat.

COROLL. VI.

8. Quoniam ergo, ex quolibet genere, una curva maximi minimive proprietate prædita invenitur; omnino ejuſmodi curvæ ſatisfacientes infinitæ invenientur, quarum quævis ita erit comparata, ut inter omnes alias eadem proprietate communi gaudentes, maximi minimive proprietate ſit prædita.

Digitized by Google

SCHOLION II.

9. Hæc omnia magis illuſtrabuntur, ſi proprietatem communem, de qua hactenus in genere ſumus locuti, definiamus. Sit igitur proprietas communis, formula longitudinem arcus curvæ exprimens, maximi minimive expreſſio autem ſit $\int Zdx$; ita ut, inter omnes curvas quæ habeant arcus eidem abſciſſæ reſpondentes inter ſe æquales, ea debeat determinari, in qua pro eadem abſciſſa fiat $\int Z\,dx$ maximum vel minimum. Manifeſtum autem eſt, non ſolum infinitas lineas curvas dari pro eadem abſciſſa longitudine æquales, verum hoc etiam infinitis modis fieri poſſe. Sit enim abſciſſa communis $= a$, ſumaturque quæcunque longitudo c major quam a, infinitæ exhiberi poterunt lineæ, tum rectæ tum curvæ, quarum ſingularum longitudo ſit $= c$; atque inter has definiri poterit una, in qua ſit $\int Z\,dx$ maximum vel minimum. Loco c autem infinitæ accipi poſſunt quantitates; eo quod alia non adeſt conditio, niſi ut ſit $c > a$; atque quilibet valor pro c aſſumtus dabit unam curvam maximi minimive proprietate præditam. Quamobrem, pro infinitis ipſius c valoribus, infinitæ reperientur lineæ curvæ quæſtioni ſatisfacientes. Neque tamen idcirco Quæſtio pro indeterminata eſt habenda: nam ſolutio ipſa, infinitas curvas ſatisfacientes præbens, ita eſt interpretanda, ut unaquæque harum curvarum inventarum inter omnes alias æque longas poſſideat valorem formulæ $\int Z\,dx$ in maximo minimove gradu. Perſpicuum autem eſt, quod hîc de æqualibus arcubus curvarum oſtendimus, idem de alia quacunque formula ſeu expreſſione indeterminata valere debere. Ita ſi, inter omnes curvas quæ, pro data abſciſſa $x = a$, valorem formulæ $\int T\,dx$ eundem continent, ea requiratur in qua ſit $\int Z\,dx$ maximum vel minimum; tum infinitæ quidem reperientur lineæ ſatisfacientes: verum hæ inter ſe ita diſcrepabunt, ut quælibet, inter omnes alias poſſibiles lineas curvas ſecum valorem formulæ $\int T\,dx$ communem habentes, contineat formulæ $\int Z\,dx$ valorem maximum vel minimum.

PRO-

Propositio I. Theorema.

10. *Quæ curva, inter omnes omnino curvas eidem abſciſſæ reſpondentes, maximi minimive cujuſpiam propoſiti proprietate gaudet; eadem curva ſimul, inter omnes curvas communi quacunque cum ipſa proprietate præditas, eadem maximi minimive proprietate gaudebit.*

Demonstratio.

Sit maximi minimive expreſſio $= A$, proprietas autem communis $= B$; eritque tam A quam B, vel formula integralis indefinita, vel expreſſio ex hujuſmodi pluribus formulis compoſita. Ponamus jam curvam eſſe inventam, quæ, inter omnes omnino curvas eidem abſciſſæ reſpondentes, expreſſionem A contineat maximam vel minimam; ea curva certum quemdam expreſſionis B continebit valorem; præter eam autem dabuntur innumerabiles aliæ, in quas idem expreſſionis B valor competet; hæque innumerabiles curvæ omnes jam continentur in illis omnibus omnino curvis, ex quibus ea, in qua expreſſio A eſt maximum minimumve, eſt inventa. Cum igitur hæc curva, inter omnes omnino curvas, propoſita maximi minimive proprietate gaudeat; eadem quoque, inter illas infinitas curvas ſecum expreſſionem B communem habentes, valorem expreſſionis A maximum minimumve poſſidebit. Q. E. D.

Coroll. I.

11. Methodus igitur abſoluta etiam Problematis Methodi relativæ reſolvendis inſervit: dum unam ſemper curvam ſatisfacientem exhibet. Verum tamen ſolutionem completam non largitur.

Coroll. II.

12. Curva ergo, quæ, inter omnes, expreſſionem A habet maxi-

maximam vel minimam, erit una ex infinitis illis curvis, quarum ſingulæ, inter omnes alias ſecum communi proprietate *B* gaudentes, eandem expreſſionem *A* maximam habent minimamve.

Coroll. III.

13. Solutio igitur Problematis, quo, inter omnes curvas eadem communi proprietate *B* præditas, ea quæritur in qua ſit *A* maximum vel minimum, latius patebit, quam ſi abſolute, inter omnes curvas, ea quæreretur in qua eſt *A* maximum vel minimum; illaque ſolutio hanc tanquam caſum ſpecialem in ſe comprehendet.

Propositio II. Problema.

14. *Methodum reſolvendi Problemata, in quibus, inter omnes curvas communi quadam proprietate gaudentes, ea requiritur quæ maximi minimive cujuſpiam propoſiti proprietate gaudeat, in genere adumbrare.*

Solutio.

Fig. 15. Omne maximum vel minimum ita eſt comparatum, ut, facta mutatione infinite parva, valor ejus omnino non immutetur. Quamobrem ſi curva az, inter omnes curvas eidem abſciſſæ AZ reſpondentes, quæ quidem communi proprietate *B* gaudeant, habeat valorem expreſſionis *A* maximum vel minimum; eundem valorem retinebit, ſi ipſi talis mutatio infinite parva inferatur, qua communis proprietas *B* non turbetur. Ad hoc autem non ſufficit, ut ante fecimus, unicam applicatam, puta Nn, particula infinite parva nν auxiſſe: quoniam enim hoc modo tota mutatio unica conditione determinatur, per eam effici nequit, ut tam proprietas communis *B* in ipſam curvam & immutatam æqualiter competat, quam maximi minimive expreſſio *A*. Quocirca mutationem adhibendam binis conditionibus

nibus determinatum eſſe oportet; id quod obtinebitur, ſi binæ applicatæ N n, & O o particulis infinite parvis n ν & o ω augeantur. Quod ſi ergo curva hoc modo immutari concipiatur; primum efficiendum eſt, ut proprietas communis cum in ipſam curvam tum in mutatam æque competat; deinde etiam maximi minimive expreſſio in utraque curva eundem valorem retinere debebit. Prius præſtabitur, ſi expreſſionis, qua proprietas communis continetur, valor differentialis inveſtigetur, oriundus ex translatione binorum n & o in ν & ω, iſque evaneſcens ponatur: poſteriori vero conditioni ſatisfiet, ſi pari modo valor differentialis expreſſionis, quæ maximum minimumve eſſe debet, quæratur, oriundus ex binis particulis n ν & o ω, atque nihilo æqualis ponatur. Hoc pacto, duæ obtinebuntur æquationes, altera ex proprietate communi, altera ex maximi minimive expreſſione; utraque autem ejuſmodi habebit formam $S.\, n\nu + T.\, o\omega = o$; in qua S & T erunt quantitates ad curvam pertinentes. Ex binis autem ejuſmodi æquationibus eliminabuntur particulæ n ν & o ω; provenietque æquatio pro curva quæſita, quæ, inter omnes alias eadem communi proprietate *B* præditas, habeat valorem expreſſionis *A* maximum vel minimum. *Q. E. I.*

COROLL. I.

15. Solutio igitur hujuſmodi Problematum quoque reducitur ad inventionem valorum differentialium: ipſi autem valores differentiales ab iis quos ante dedimus in hoc diſcrepant, quod ex translatione duorum curvæ punctorum definiri debeant.

COROLL. II.

16. Ejuſmodi valores differentiales ergo ex duabus particulis n ν & o ω oriundos, in quovis Problemate, binos inveſtigari oportet; alterum pro proprietate communi, alterum pro maximi minimive expreſſione.

Digitized by Google

Coroll. III.

17. Inventis autem in quovis Problemate his duobus valoribus differentialibus, uterque nihilo æqualis poni debet; ex quo binæ naſcentur æquationes, quæ, eliminandis particulis aſſumtis n ξ & o ω, præbebunt unam æquationem naturam curvæ quæſitæ exprimentem.

Coroll. IV.

18. Si ergo, inter omnes curvas eidem abſciſſæ reſpondentes, quæ communi proprietate *B* æqualiter ſunt præditæ, ea requiratur, in qua expreſſio *A* fiat maximum vel minimum; tum utriuſque expreſſionis *A* & *B* valores differentiales, ex binis particulis n ν & o ω oriundi, quæri, & nihilo æquales poni debent; ex quibus duabus æquationibus ſi eliminentur particulæ n ν & o ω, emerget æquatio pro curva quæſita.

Coroll. V.

19. In hac itaque operatione, ambæ expreſſiones *A* & *B* omnino pariter tractantur; neque in conſiderationem venit, utra vel proprietatem communem vel maximum minimumve denotet. Ex quo perſpicuum eſt, eandem ſolutionem prodire debere, ſi expreſſiones *A* & *B* inter ſe commutentur.

Coroll. VI.

20. Eadem ergo ſolutio locum habebit, ſive, inter omnes curvas communi proprietate *B* gaudentes, ea quæratur in qua ſit *A* maximum vel minimum: ſive viciſſim, inter omnes curvas communi proprietate *A* gaudentes, ea quæratur in qua ſit *B* maximum vel minimum.

SCHO-

Digitized by Google

SCHOLION.

21. Ambas expreſſiones A & B, licet in ſe ſpectatæ res omnino diverſas ſignificent, inter ſe commutabiles eſſe ipſa ſolutionis natura ſponte patet. Quod ſi enim ad binas particulas $n\nu$ & $o\omega$ reſpiciamus, quibus applicatæ Nn & Oo augentur; primum eas ita comparatas eſſe oportet, ut proprietas communis B, tam in ipſa curva quam in mutata, eundem valorem obtineat; ſcilicet proprietas communis B in curvam $amnopz$ & in $am\nu\omega pz$ æque competere debet: deinde pari modo per eaſdem particulas $n\nu$ & $o\omega$ efficiendum eſt, ut expreſſio A, quæ maximum minimumve eſſe debet, tam pro curva $amnopz$ quam pro $am\nu\omega pz$ eundem valorem recipiat. Atque adeo, tam proprietas communis, quam maximi minimive natura, eandem plane conditionem in calculum inducit; ex quo manifeſtum eſt ambas expreſſiones datas, quarum altera proprietatem communem, altera maximi minimive rationem continet, inter ſe commutari atque confundi poſſe, ſalva Solutione. Hanc ob rem ergo, in Solutione hujuſmodi Problematum, ſufficit noſſe ambas illas expreſſiones; neque ad Solutionem abſolvendam noſſe opus eſt, utra proprietatem communem aut maximum minimumve ſignificet. Sic ſi, inter omnes curvas longitudine æquales, quæratur ea, quæ maximam aream comprehendat; eadem reperitur curva quæ prodit, ſi, inter omnes curvas æquales areas includentes, ea quæratur quæ ſit breviſſima, vel minimam longitudinem habeat. Hæc ita ſe habent, ſi maximi minimive quod quæritur natura ita fuerit comparata, ut ejus valor differentialis ſit $=0$. Jam ſupra autem animadvertimus, duplicis generis dari maxima & minima, in quorum altero valor differentialis ſit $=0$, in altero vero $=\infty$. Hic vero tantum maxima ac minima prioris generis contemplamur; nam, in hac Methodo relativa, poſterius genus locum omnino habere nequit. Quod ſi enim valor differentialis, qui convenit maximi minimive expreſſioni, infinite magnus ponatur; tum ex hoc ſolo æquatio pro curva reperitur; neque ideo proprietas communis in computum ingreditur. Quare, ſi hujus

 generis

generis maximum vel minimum in Methodo abſoluta locum habet, eadem curva in Methodo relativa eadem proprietate gaudebit, quæcunque proprietas communis adjungatur. Cum igitur totum Solutionis hujuſmodi Problematum momentum verſetur in inventione valorum differentialium, qui ex binis particulis n ν & o ω oriuntur; Methodum trademus, ejuſmodi valores differentiales pro quacunque expreſſione indeterminata inveniendi, eo modo, quo ſupra uſi ſumus ad inveniendos valores differentiales ex unica particula n ν oriundos.

PROPOSITIO III. PROBLEMA.

Fig. 15. 22. *Propoſita quacunque expreſſione indeterminata, quæ ad datam abſciſſam* A Z *referatur; invenire ejus valorem differentialem, ortum ex translatione binorum curvæ punctorum* n *&* o *in* ν *&* ω.

SOLUTIO.

Ponamus abſciſſam A I $= x$, & applicatam I i $= y$, erit K k $= y'$, L l $= y''$, M m $= y'''$, N n $= y^{\text{IV}}$, O o $= y^{\text{V}}$, P p $= y^{\text{VI}}$ &c. Harum applicatarum duæ tantum, nempe y^{IV} & y^{V} patiuntur alterationem a particulis n ν & o ω ipſis adjunctis. Erit igitur applicatæ y^{IV} valor differentialis $=$ n ν, & applicatæ y^{V} valor differentialis $=$ o ω, reliquarum vero applicatarum omnium valor differentialis erit $= 0$. Hinc reliquarum quantitatum ad curvam pertinentium p, q, r, s, &c. valores differentiales habebuntur, quatenus eæ ab his binis applicatis y^{IV} & y^{V} pendent. Sic cum ſit $p = \frac{y' - y}{dx}$, erit valor differentialis ipſius $p = 0$; ſimiliterque ipſius p', & p'': at cum ſit $p''' = \frac{y^{\text{IV}} - y'''}{dx}$, erit ipſius p''' valor differentialis $= \frac{n\nu}{dx}$; &, ob $p^{\text{IV}} = \frac{y^{\text{V}} - y^{\text{IV}}}{dx}$, erit valor differentialis ipſius $p^{\text{IV}} = \frac{o\omega}{dx} - \frac{n\nu}{dx}$; porroque ipſius p^{V} erit $= -\frac{o\omega}{dx}$. Deinde cum ſit

$q =$

Digitized by Google

$q = \frac{p' - p}{dx}$; erit valor differentialis ipſius $q'' = \frac{n\nu}{dx^2}$; ipſius $q''' = \frac{o\omega}{dx^2} - \frac{2n\nu}{dx^2}$, ipſius $q'^{V} = - \frac{2o\omega}{dx^2} + \frac{n\nu}{dx^2}$; ipſius $q^{V} = \frac{o\omega}{dx^2}$. Hocque modo ſimiliter progredi licet ad ſequentes quantitates r, s, &c. cum ſuis derivativis; hincque naſcetur ſequens Tabella, qua ſingularum harum quantitatum valores differentiales exhibentur.

$d.\ y'^{V} = n\nu$	$d.\ q'' = \frac{n\nu}{dx^2}$
$d.\ y^{V} = o\omega$	
$d.\ p''' = \frac{n\nu}{dx}$	$d.\ q''' = - \frac{2n\nu}{dx^2} + \frac{o\omega}{dx^2}$
$d.\ p'^{V} = - \frac{n\nu}{dx} + \frac{o\omega}{dx}$	$d.\ q'^{V} = \frac{n\nu}{dx^2} - \frac{2o\omega}{dx^2}$
$d.\ p^{V} = - \frac{o\omega}{dx}$	$d.\ q^{V} = \frac{o\omega}{dx^2}$
$d.\ r' = + \frac{n\nu}{dx^3}$	$d.\ s = + \frac{n\nu}{dx^4}$
$d.\ r'' = - \frac{3n\nu}{dx^3} + \frac{o\omega}{dx^3}$	$d.\ s' = - \frac{4n\nu}{dx^4} + \frac{o\omega}{dx^4}$
$d.\ r''' = + \frac{3n\nu}{dx^3} - \frac{3o\omega}{dx^3}$	$d.\ s'' = + \frac{6n\nu}{dx^4} - \frac{4o\omega}{dx^4}$
$d.\ r'^{V} = - \frac{n\nu}{dx^3} + \frac{3o\omega}{dx^3}$	$d.\ s''' = - \frac{4n\nu}{dx^4} + \frac{6o\omega}{dx^4}$
$d.\ r^{V} = - \frac{o\omega}{dx^3}$	$d.\ s'^{V} = + \frac{n\nu}{dx^4} - \frac{4o\omega}{dx^4}$
	$d.\ s^{V} = + \frac{o\omega}{dx^4}$

&c.

Ex hac Tabella perſpicitur, in valoribus differentialibus totidem terminos particula $o\omega$ affectos occurrere, ac particula $n\nu$; atque in utriſque pares adeſſe coefficientes: diſcrimen vero in hoc conſiſtere, ut cuilibet termino particula $o\omega$ affecto reſpon-

 deat

deat quantitas immediate ſequens eam, cui reſpondet ſimilis terminus particula $n\nu$ affectus. Sic dum terminus $-\frac{2n\nu}{dx^2}$ reperitur in valore differentiali quantitatis q''', ita terminus $-\frac{2o\omega}{dx^2}$ adeſt in valore differentiali quantitatis ſequentis q^{V}. Deinceps, ob duplicis generis terminos in valoribus differentialibus occurrentes, quorum alteri particulam $n\nu$, alteri particulam $o\omega$ involvunt, valor differentialis cujuſcunque expreſſionis indeterminatæ hujuſmodi habebit formam $n\nu.\ I+o\omega.\ K$; de qua primum, manifeſtum eſt membrum prius $n\nu.\ I$ eſſe ejuſdem expreſſionis valorem differentialem, qui oritur ſi ſola particula $n\nu$ conſideretur; eritque ideo $n\nu.\ I$ ille ipſe valor differentialis, quem ſupra pro quavis expreſſione oblata definire docuimus; ita ut hoc membrum per præcepta ſupra tradita pro quavis expreſſione indeterminata exhibere liceat. Quod ad alterum membrum $o\omega.\ K$ attinet, quia ſinguli termini in quibus $o\omega$ ineſt perpetuo reſpondent quantitatibus ſequentibus eas, quibus reſpondent ſimiles termini particulam $n\nu$ involventes; palam eſt quantitatem K fore valorem, quem quantitas I in proximo ſequente loco induit, atque idcirco eſſe $K=I'=I+dI$. Quare cum membrum $n\nu.\ I$ ex præceptis jam ſupra datis aſſignare queamus, ex eo porro alterum membrum $o\omega.\ K=o\omega\,(I+dI)$ innoteſcet. Sit igitur V expreſſio quæcunque indeterminata, cujus valorem differentialem ex duabus particulis $n\nu$ & $o\omega$ oriundum definiri oporteat. Ponamus ejus valorem differentialem ex unica particula $n\nu$ oriundum eſſe $=n\nu.\ I$; eritque valor differentialis, qui ex ambabus particulis $n\nu$ & $o\omega$ oritur, $=n\nu.\ I+o\omega.\ I'$, ſeu $=n\nu.\ I+o\omega\,(I+dI)$; qui igitur ope regularum ſupra datarum facile aſſignari poterit. *Q. E. I.*

Coroll. I.

23. Omnium ergo expreſſionum, quarum valores differentiales ex unica particula $n\nu$ oriundos invenire docuimus, earundem valo-

Digitized by Google

valores differentiales ex binis particulis $n\nu$ & $o\omega$ oriundos nunc definire poſſumus.

COROLL. II.

24. Hæc igitur Methodus valebit tam ad expreſſionum valores differentiales inveniendos, qui non pendent a quantitate abſciſſæ propoſitæ AZ, quam qui ab iſtius abſciſſæ longitudine pendent.

COROLL. III.

25. Quin etiam ſi expreſſio propoſita, quæ vel proprietatem communem continet, vel maximum minimumve eſſe debet, fuerit functio duarum pluriumve formularum integralium; ejus valor differentialis ex binis particulis $n\nu$ & $o\omega$ oriundus eadem lege definietur.

SCHOLION.

26. In Capitibus ſuperioribus vidimus valorem differentialem cujuſcunque expreſſionis, qui ex unica particula $n\nu$ oritur, hujuſmodi habere formam $n\nu . dx . T$; ſeu $n\nu . T dx$; ubi T denotat quantitatem finitam: quare ejuſdem expreſſionis valor differentialis ex binis particulis $n\nu$ & $o\omega$ ortus erit $= n\nu . T dx + o\omega . T' dx$. quemadmodum in Solutione oſtendimus. Eadem autem forma facile ad hunc modum poteſt evinci: Scilicet ſi ponatur $o\omega = o$, tum prodire debet ipſe valor differentialis ex unica particula $n\nu$ ortus, quem ſupra invenire docuimus, eritque $n\nu . T' dx$. Sin autem ponatur $n\nu = o$, ac ſola particula $o\omega$ conſideratur, valor differentialis ſimili modo reperietur quo ſupra uſi ſumus: non autem erit $= o\omega . T dx$; nam quia particula $o\omega$ in ſitu ſequente accipitur, loco T ejus valor ſequens pariter ſumi debet; ita ut valor differentialis verus futurus ſit $= o\omega . T' dx$. Quod ſi ergo utraque particula $n\nu$ & $o\omega$ conjunctim conſideretur, erit valor differentialis $= n\nu . T dx + o\omega . T' dx$; eo quod in ipſo cal-

Digitized by Google

calculo particulæ $n\nu$ & $o\omega$ nuſquam inter ſe permiſcentur, ſed utraque perpetuo ſeorſim tractari poſſit. Ut autem hæc ad notandi modum in ſuperiori capite receptum accommodemus; ponamus V eſſe expreſſionem quamcunque indeterminatam, quæ, pro abſciſſa definita $AZ = a$; valorem recipiat $= A$; ejuſque valorem differentialem ex particula $n\nu$ ortum eſſe $= n\nu . dA$; ubi dA nobis denotet idem quod ante Tdx; poteritque iſte valor dA ex expreſſione V, modo in Capitibus præcedentibus expoſito, inveniri. Hoc invento erit ejuſdem expreſſionis V valor differentialis ex binis particulis $n\nu$ & $o\omega$ oriundus $= n\nu . dA + o\omega . dA'$ ubi dA' denotat valorem dA ſuo differentiali auctum. Quanquam autem iſta valorum differentialium ex binis particulis oriundorum ad noſtrum inſtitutum omnino eſt neceſſaria; tamen ſolutio ipſa Problematum huc pertinentium eo iterum reducetur, ut per ſolos valores differentiales modo ſupra expoſito inventos, qui ſcilicet ex unica particula $n\nu$ naſcuntur, abſolvi queat; id quod ex ſequente Propoſitione mox patebit.

PROPOSITIO IV. PROBLEMA.

27. *Inter omnes curvas ad eandem datam abſciſſam* $AZ = a$ *relatas, in quas idem valor expreſſionis indefinitæ* W *competit; determinare eam, in qua ſit expreſſio* V *maximum vel minimum.*

SOLUTIO.

Ponamus curvam az quæſito ſatisfacere, atque expreſſionem W in ea obtinere valorem determinatum $= B$; erit ergo hæc curva az inter omnes alias curvas ad eandem abſciſſam AZ relatas, in quibus expreſſio W eundem obtinet valorem, ita comparata, ut in ea expreſſio V maximum minimumve valorem recipiat, qui ſit $= A$. Ad curvam ergo hanc inveniendam, poſitis abſciſſa indefinita $AI = x$, & applicata reſpondente $Ii = y$; binæ applicatæ Nn & Oo particulis infinite parvis $n\nu$ & $o\omega$ augeri concipiantur: quo facto, tam ipſius W quam ipſius V valor

lor differentialis, qui ex his duabus particulis $n\nu$ & $o\omega$ adjunctis naſcetur nihilo æqualis poni debebit, uti in Propoſitione ſecunda oſtendimus. Sit jam expreſſionis V valor differentialis ex unica particula $n\nu$ ortus $= n\nu.\, dA$, atque expreſſionis alterius W valor differentialis ex eadem unica particula $n\nu$ ortus $= n\nu.\, dB$, quos valores differentiales ex præceptis in ſuperioribus Capitibus datis invenire licebit. Nunc igitur, dum binas particulas $n\nu$ & $o\omega$ contemplamur, erit expreſſionis V valor differentialis $= n\nu.\, dA + o\omega.\, dA'$; alterius vero expreſſionis W valor differentialis erit $= n\nu.\, dB + o\omega.\, dB'$. Quocirca, ad quæſitam curvam inveniendam, fieri oportet cum $n\nu.\, dA + o\omega.\, dA' = o$. tum etiam $n\nu.\, dB + o\omega.\, dB' = o$. Multiplicentur ambæ æquationes per quantitates quaſcunque, ita ut prodeat

$$n\nu.\, \alpha dA + o\omega.\, \alpha dA' = o$$
$$n\nu.\, \beta dB + o\omega.\, \beta dB' = o.$$

Fiatque ad particulas $n\nu$ & $o\omega$ eliminandas tam $\alpha dA + \beta dB = o$, quam $\alpha dA' + \beta dB' = o$; eruntque α & β ejuſmodi quantitates, ſive conſtantes, ſive variabiles, quæ utrique æquationi ſatisfaciunt. Quoniam vero eſt $\alpha dA + \beta dB = o$, erit quoque $\alpha' dA' + \beta' dB' = o$; quæ æquatio, cum $\alpha dA' + \beta dB' = o$ comparata, monſtrat eſſe debere $\alpha' = \alpha$, & $\beta' = \beta$; ex quo quantitates hæ α & β debebunt eſſe conſtantes, & quidem quæcunque. Sumtis itaque pro α & β quantitatibus quibuſcunque conſtantibus, æquatio pro curva erit $\alpha dA + \beta dB = o$. Hæc eadem æquatio prodit, ſi methodo conſueta particulas $n\nu$, & $o\omega$ eliminemus. Erit nempe $\frac{n\nu}{o\omega} = -\frac{dA'}{dA} = -\frac{dB'}{dB}$, ideoque $\frac{dA'}{dA} = \frac{dB'}{dB}$, ſeu $\frac{ddA}{dA} = \frac{ddB}{dB}$, ob $dA' = dA + ddA$, & $dB' = dB + ddB$. Æquatio autem $\frac{ddA}{dA} = \frac{ddB}{dB}$ integrata dat $l\,dA \pm l\,dB + lC$. Seu $dA =$

$C\,dB$; quæ, poſito $C = -\frac{\mathfrak{C}}{\alpha}$, tranſit in $\alpha\,dA + \mathfrak{C}\,dB = 0$; quam ipſam ante invenimus. Quamobrem ad Problema reſolvendum oportet, tam expreſſionis proprietatem communem continentis W, quam expreſſionis quæ maximum minimumve eſſe debet V, valores differentiales, methodo in ſuperioribus Capitibus tradita, inveſtigare, eoſque per quantitates conſtantes quaſcunque multiplicare, ſummamque $= 0$ ponere; quo facto, reſultabit æquatio naturam curvæ quæſitæ exprimens. *Q. E. I.*

COROLL. I.

28. Nunc igitur, ad Quæſtiones in hac Propoſitione contentas reſolvendas, ſufficit noſſe valores differentiales ex unica particula $n\nu$ oriundos; quos ſupra jam expedite invenire docuimus.

COROLL. II.

29. Quare ad hoc negotium in ſubſidium vocari debebit Caput præcedens IV, ex eoque cum §. 7 tum §. 31. In loco priore enim continentur præcepta valores differentiales inveniendi, ſi expreſſiones indeterminatæ propoſitæ fuerint formulæ integrales ſingularis, in altero vero, ſi ſint functiones duarum pluriumve ejuſmodi formularum integralium.

COROLL. III.

30. Propoſita ergo proprietate communi W, & maximi minimive expreſſione V, utriuſque expreſſionis valorem differentialem ex his præceptis quæri oportet: quibus inventis, & per conſtantes arbitrarias mulplicatis, eorum aggregatum nihilo æquale poſitum dabit æquationem pro curva quæſita.

COROLL. IV.

31. Si, inter omnes omnino curvas eidem abſciſſæ A Z reſpon-

ſpondentes, quæratur ea, in qua expreſſio V maximum minimumve obtineat valorem; pro ea habetur iſta æquatio $dA = 0$; denotante dA valorem differentialem expreſſionis V.

COROLL. V.

32. Quodſi autem, inter omnes curvas eidem abſciſſæ AZ reſpondentes, in quas expreſſio W æqualiter competat, quæratur ea in qua expreſſio V maximum minimumve habeat valorem; invenitur pro ea iſta æquatio $\alpha dA + \beta dB = 0$.

COROLL. VI.

33. Perſpicuum ergo eſt, curvam, quæ, inter omnes omnino curvas, habeat V maximum vel minimum, cujus æquatio eſt $dA = 0$, contineri in æquatione $\alpha dA + \beta dB = 0$, qua exprimitur curva, quæ, inter omnes eadem communi proprietate W gaudentes, habeat V maximum vel minimum.

COROLL. VII.

34. In ipſa igitur prima æquatione, quam Solutio præbet, $\alpha dA + \beta dB = 0$; jam ineſt una conſtans arbitraria; quæ autem per id determinari debet, ut valor expreſſionis W datum obtineat valorem.

COROLL. VIII.

35. Problema itaque ſic ſolvi poterit, ut, inter omnes curvas eidem abſciſſæ AZ reſpondentes, in quibus expreſſio W eundem datum obtineat valorem, definiatur ea in qua ſit valor ipſius V maximus vel minimus.

COROLL. IX.

36. Ex his denique intelligitur, Solutionem Problematis propoſiti

poſiti, convenire cum Solutione hujus Problematis, quo, inter omnes omnino curvas eidem abſciſſæ AZ reſpondentes, requiratur ea quæ habeat $\alpha V + \beta W$ maximum vel minimum. Quæ quæſtio, etſi ad Methodum abſolutam pertineat, tamen dat æquationem $\alpha dA + \beta dB = 0$, quam ipſam invenimus.

SCHOLION I.

37. Ex his igitur non ſolum Methodus facilis atque expedita colligitur, omnes Quæſtiones huc pertinentes reſolvendi; verum etiam natura hujus generis Problematum penitius cognoſcitur. Primo enim apparet, quod jam ſupra demonſtravimus, Solutionem eandem fore, ſive, inter omnes curvas communi proprietate W præditas, quæratur ea quæ habeat V maximum vel minimum; ſive inverſe, inter omnes curvas communi proprietate V præditas, ea requiratur in qua ſit W maximum vel minimum. Deinde etiam intelligitur, Quæſtionem ita proponi poſſe, ut ejus Solutio ad Methodum maximorum ac minimorum abſolutam pertineat; congruit enim Problema propoſitum cum hoc, quo, inter omnes omnino curvas ad eandem abſciſſam AZ relatas, requiritur ea in qua ſit iſta expreſſio $\alpha V + \beta W$ maximum vel minimum; atque hæc Problematis transformatio in cauſa eſt, quod Solutio per valores differentiales ex unica particula $n\nu$ oriundos perfici queat, neque amplius opus ſit duas hujuſmodi particulas conſiderare, prouti primo intuitu natura Quæſtionis poſtulare videbatur. Hanc autem convenientiam poſtmodum, per ſe, ac ſine iſta Methodo qua binæ particulæ conſiderantur, demonſtrabimus; quo veritas hæc, ſummi in iſto negotio momenti, magis confirmetur. Ad ſolvendas cæterum hujuſmodi Quæſtiones, ante oculos habere oportet præcepta Capite præcedente in compendium redacta; quorum ope valores differentiales quarumcunque expreſſionum inveniri poterunt. Primo enim, §. 7 illius Capitis recenſentur Caſus, quibus formularum integralium ſolitariarum valores exhibentur: tum vero §. 31 traditur Methodus inveniendi valores differentiales expreſſio-

preſſionum, quæ ex duabus pluribuſve formulis integralibus utcunque ſint compoſitæ. Ex his itaque ſubſidiis, pro quavis Quæſtione oblata, tam maximi minimive expreſſionis quam proprietatis communis valor differentialis aſſignari poterit: utroque autem invento, æquatio pro Curva quæſita nullo negotio formabitur; cum tantum opus ſit aggregatum quorumcunque multiplorum illorum binorum valorum differentialium nihilo æquale poni. Hæcque æquatio inventa, deinceps pari modo erit tractanda, quo ſupra, cum in reductione ad conſtruendum, tum in integratione uſi ſumus.

SCHOLION II.

38. Jam obſervavimus in æquatione $\alpha\,dA + \mathfrak{C}\,dB = 0$, quam Solutio immediate ſuppeditat, unam ineſſe quantitatem conſtantem; quæ autem non omnino ſit arbitraria, ſed ex conditione propoſita debeat determinari. Scilicet, cum in omnes curvas ex quibus quæſitam definiri oportet, eadem expreſſio W æqualiter competere debeat, ſeu in omnibus eundem valorem, puta B, obtinere; hæc quantitas B tanquam data ſpectari poteſt; atque cum ipſa in calculum non ingrediatur, ita conſtantes α & $\mathfrak{C}$ definire licebit, ut valor expreſſionis W, abſciſſæ $AZ = a$ reſpondens, ipſi B æqualis fiat; hocque pacto, Quæſtio alioquin indeterminata determinabitur. Eatenus autem tantum determinabitur, quatenus, per integrationes poſt inſtituendas, novæ conſtantes arbitrariæ etiam per totidem puncta definiuntur. Prorſus nimirum ut ante, totidem puncta præſcribi poterunt, per quæ curva quæſita tranſeat, quot novæ conſtantes per integrationes ingredi cenſendæ ſunt. Horum autem numerus innoteſcet ex gradu differentialium ſummo, qui in æquatione inerit. Quoniam vero tota Quæſtio ad Methodum abſolutam revocari poteſt, numerus iſtiuſmodi conſtantium perpetuo erit par; ſeu æquatio reſultans $\alpha\,dA + \mathfrak{C}\,dB = 0$, erit vel finita, vel differentialis ſecundi gradus, vel differentialis quarti gradus, vel differentialis ſexti gradus, vel octavi, vel ita porro. Quod

ſi æquatio prodit finita, tum quoque curva penitus jam erit determinata; ſiquidem ratio inter α & $\mathfrak{C}$ ita definiatur, ut expreſſio W datum recipiat valorem B in curva inventa, quam determinationem perpetuo adhiberi ponimus. Si æquatio inveniatur differentialis ſecundi gradus, tum duobus punctis curva inventa determinabitur; congruum autem ac more receptum eſt ipſos curvæ terminos a & z præſcribi, hiſque caſibus Problema determinabitur, ſi conditio iſta adjungatur, ut curva quæſita intra datos terminos a & z contineatur. Sin autem æquatio prodeat differentialis quarti gradus, tum quatuor punctis pro lubitu aſſignatis, curva ſatisfaciens determinabitur; hæc igitur definiri ita conveniet, ut, præter terminos extremos a & z, ſimul poſitio tangentium in his terminis præſcribatur. Sin perveniatur ad æquationem differentialem ſexti gradus, tum curva per ſex quæcunque puncta determinabitur: eorum autem loco præſcribi poterunt primo ambo termini a & z, tum poſitio tangentium in his terminis, ac tertio curvedo in his ipſis locis ſeu radii oſculi quantitas. His igitur notatis, intelligetur ex ipſa Solutione cujuſmodi conditio ad Problematis cujuſque propoſitionem adjungi debeat, ut id fiat penitus determinatum: hæcque admonitio, non ſolum hîc, ſed etiam in Methodo abſoluta atque reliqua Methodo relativa, locum habet.

SCHOLION III.

39. Diſcrimen hîc etiam maximi momenti inprimis eſt notandum, ex quo in Methodo abſoluta primariam tractationis partitionem deſumſimus. Conſiſtit id autem in modo, quo curva inventa Quæſtioni ſatisfacit. Fieri enim poteſt, ut ejus quæcunque portio ad abſciſſam indefinitam relata requiſita proprietate gaudeat; deinde etiam dantur caſus, quibus nonniſi ea portio quæ definitæ abſciſſæ $AZ = a$ reſpondet, conditioni Problematis ſatisfaciat. Illud ſcilicet evenit, ſi quantitas hæc a in æquationem, quam Solutio ſuppeditat, vel omnino non ingreditur, vel in quantitates arbitrarias α & $\mathfrak{C}$ comprehendi queat.

queat. Ex quo manifeſtum eſt, ſi ambæ formulæ W & V in caſu primo §. 7 Capitis præcedentis recenſito contineantur, tum curvæ inventæ quamlibet portionem ad Quæſtionem eſſe accommodatam. Deinde vero etiam fieri poteſt, ut licet quantitas a, ſeu quantitates ab ea pendentes, vel in alterutro valore differentiali inſint, vel in utroque; tamen eæ vel ſe mutuo tollant in æquatione $\alpha\, dA + \mathcal{C}\, dB = 0$, vel ſub arbitrariis α & $\mathcal{C}$ comprehendi queant; quo caſu pariter quamvis curvæ inventæ portionem ſatisfacere oportet. Hoc autem tantum locum habet, ſi non datus ac determinatus præſcribatur valor, quem proprietas communis W in portione ſatisfaciente obtinere debeat: tum enim fieri nequit, ut in quavis portione eundem valorem ſortiatur. Ex Solutione autem unius cujuſque Quæſtionis facile intelligetur, qua conditione, ſive tota curva a z, ſive quævis portio, ſatisfacere queat; id quod commodiſſime in Exemplis oſtendi poterit.

EXEMPLUM I.

40. *Inter omnes curvas ad abſciſſam* AZ *relatas, in quibus formula* ſ y x d x *eundem obtinet valorem, invenire eam in qua ſit valor formulæ* ſ y y d x *minimus.*

Erit igitur proprietas communis $W = \int xy\,dx$, cujus, ob $d\,xy = y\,dx + x\,dy$, valor differentialis eſt $= n\nu.\ dx.\ x$. Maximi autem minimive formula eſt $V = \int yy\,dx$, cujus, ob $d.\ yy = 2y\,dy$, valor differentialis eſt $= n\nu.\ dx.\ 2y$. Obtinebitur ergo, diviſione per $n\nu.\ dx$ inſtituta, hæc æquatio $\alpha x + 2\mathcal{C}y = 0$; ex qua patet Quæſtioni ſatisfacere lineam rectam in A cum axe AZ angulum quemcunque conſtituentem. Et quia longitudo abſciſſæ $AZ = a$ non in computum ingreditur, quævis hujus rectæ portio æque ſatisfaciet. Quod ſi autem poſtuletur, ut pro data abſciſſa $AZ = a$, formula $\int y\,x\,dx$ datum obtineat valorem, puta B; tum ob $y = m\,x$, fiet $\int y\,x\,dx = \frac{1}{3}mx^3$; ideoque $\frac{1}{3}ma^3 = B$; ex quo poſitio lineæ rectæ ita

ita definietur, ut eſſe debeat $y = \frac{3Bx}{a^3}$. Hæc igitur recta jam iſta proprietate gaudebit, ut ea inter omnes lineas, ſive rectas, ſive curvas, quæ pro data abſciſſa A Z $= a$ habeant formulæ $\int xy\,dx$ valorem $= B$, producat formulæ $\int yy\,dx$ minimum valorem.

Exemplum II.

41. *Inter omnes curvas ejuſdem longitudinis puncta* a *&* z *jungentes, invenire eam quæ maximam vel minimam aream* a A Z z *comprehendat.*

Quoniam proprietas communis eſt longitudo arcus $= \int dx \surd(1+pp)$; erit ejus valor differentialis $= -$ nν. $d. \frac{p}{\surd(1+pp)}$. Deinde maximi minimive formula eſt $\int y\,dx$, cujus valor differentialis eſt nν. dx: unde pro curva quæſita iſta habebitur æquatio $dx = b\,d. \frac{p}{\surd(1+pp)}$, & integrando $x + c = \frac{bp}{\surd(1+pp)}$; ideoque $p = \frac{x+c}{\surd(b^2-(x+c)^2)} = \frac{dy}{dx}$. Hinc ergo integrando fit $y = f \pm \surd(b^2-(x+c)^2)$, ſeu $b^2 = (y-f)^2 + (x+c)^2$; quæ eſt æquatio generalis pro Circulo. Quamobrem arcus Circuli quicunque per puncta a & z ductus, inter omnes alias lineas curvas ejuſdem longitudinis, vel maximam vel minimam aream a A Z z includet. Duplici autem modo Circuli arcus datæ longitudinis intra terminos a & z conſtitui poteſt; altero, quo concavitatem axi A Z obvertit; altero, quo convexitatem. Priori caſu manifeſtum eſt aream fore maximam, poſteriore vero minimam. Atque hinc ſi dentur termini a & z, una cum longitudine curvæ intra hos terminos conſtitutæ, quam majorem quidem eſſe oportet lineam rectam hos terminos jungentem; Solutio penitus erit determinata: arcus Circuli enim hujus longitudinis per hos terminos poterit deſcribi unicus,

Digitized by Google

unicus, qui, prout vel concavitatem vel convexitatem axi AZ obvertat, aream formabit vel maximam vel minimam.

COROLLARIUM.

42. Hinc etiam patet arcum circularem az, per terminos a & z ductum, non ſolum maximam aream aAZz, inter omnes alias lineas ejuſdem longitudinis formare; ſed etiam quæcunque linea aCEDz a termino a ad terminum z ducta detur, cum ea arcus circularis az maximam includet aream. Nam ſi area aAZz eſt maxima, erit quoque area aAZz — aAC — zZD + CED, ob areas aAC, zZD & CED conſtantis magnitudinis quæcunque linea pro az capiatur, maxima. *Fig. 16.*

EXEMPLUM III.

43. *Inter omnes curvas ejuſdem longitudinis puncta* A *&* M *jungentes, invenire eam quæ, cum rectis* AC *&* MC *ad punctum fixum* C *ductis, maximam vel minimam comprehendat aream* ACM. *Fig. 7.*

Quoniam, ob data puncta A, C, M, rectæ AC & MC poſitione dantur, ponatur angulus $ACM = x$, ſeu deſcripto centro C, radio $CB = 1$, arcu circulari BS, ſit hic arcus $BS = x$, & ponatur $CM = y$; erit $Ss = dx$, $Mn = y\,dx$, & area $ACM = \frac{1}{2}\int yy\,dx$. Porro ob $mn = dy$, erit $Mm = \surd(y^2 dx^2 + dy^2) = dx\surd(yy + pp)$, poſito $dy = p\,dx$. Quare inter omnes æquationes relationem ipſarum x & y continentes, quæ, pro dato ipſius x valore, eandem præbent quantitatem $\int dx \surd(yy + pp)$, eam definiri oportet, quæ, pro eodem ipſius x valore, præbeat formulæ $\frac{1}{2}\int yy\,dx$ quantitatem vel maximam vel minimam. Cum igitur formulæ $\int dx \surd(yy + pp)$ valor differentialis ſit $= nv.\,dx\left(\frac{y}{\surd(yy+pp)} - \frac{1}{dx}\,d.\frac{p}{\surd(yy+pp)}\right)$ & formulæ $\int yy\,dx$ valor differentialis $=$

 nv.

Multiplicetur hæc per p, ut prodeat $2ydy = bdy\surd(1+pp) - bpd.\frac{yp}{\surd(1+pp)} = bd.y\surd(1+pp) - \frac{bypdp}{\surd(1+pp)} - bpd.\frac{yp}{\surd(+pp)}$, cujus integrale eſt $yy = by\surd(1+pp) - \frac{bypp}{\surd(1+pp)} - bc = \frac{by}{\surd(1+pp)} + bc$. Erit ergo $by = (yy - bc)\surd(1+pp)$, & $p = \frac{\surd(b^2y^2 - (yy-bc)^2)}{yy-bc}) = \frac{dy}{dx}$. Quare erit $dx = \frac{(yy-bc)dy}{\surd(bbyy-(yy-bc)^2}$. De hac æquatione primo notandum eſt, ſi fuerit $c = 0$ fore $dx = \frac{ydy}{\surd(bb-yy)}$; ideoque curvam eſſe Circulum, cujus centrum in axe A Z ſit poſitum; ille igitur arcus circularis, centro in axe A N ſumpto deſcriptus & per data duo puncta a & z tranſiens Quæſtioni ſatisfaciet; erit autem is unicus, ideoque ſolidum definitæ ſuperficiei generabit. Quare ſi, inter omnes curvas quæ ſolida alius atque diverſæ ſuperficiei generant, quæratur ea quæ maximum volumen producat, ea non erit Circulus, ſed alia curva in æquatione $dx = \frac{(yy-bc)dy)}{\surd\ bbyy - (yy-bc)^2)}$ contenta. Non ſolum enim, ob binas conſtantes b & c, effici poteſt, ut curva per præſcripta duo puncta a & z tranſeat; ſed etiam ut longitudo curvæ a z exiſtat datæ magnitudinis. Cæterum longitudo curvæ, ob $\int dx\surd(1+pp) = \int\frac{bydx}{yy-bc}$ fiet $= \int\frac{bydy}{\surd(bbyy-(yy-bc)^2)}$; cujus integrale a quadratura Circuli pendet, eſtque $= \frac{b}{2}$ A cof. $\frac{b(2c+b) - 2yy}{b\surd(bb+4bc)} +$ *Conſt.* Quod ſi autem b ponatur $= \infty$, caſus oritur ſingularis; æquatio namque prodit hæc $dx = -\frac{cdy}{\surd(yy-cc)}$, quæ eſt pro curva Catenaria convexitatem axi A Z obvertente.

Digitized by Google

EXEMPLUM V.

45. *Inter omnes curvas* az *æquales areas* aAZz *continentes, determinare eam, quæ circa axem* AZ *rotata generet ſolidum minimæ ſuperficiei.*

Quoniam proprietas communis in area $= \int y\,dx$ conſtituitur; erit ejus valor differentialis $= nv.\,dx$. Deinde formula, quæ minimum eſſe debet, eſt $\int y\,dx\,\surd(1+pp)$, cujus valor differentialis eſt $= nv.\,(dx\,\surd(1+pp) - d.\,\frac{yp}{\surd(1+pp)})$; unde orietur, pro curva quæſita, iſta æquatio $n\,dx = dx\surd(1+pp) - d.\,\frac{yp}{\surd(1+pp)}$, quæ, per p multiplicata & integrata, præbet $ny + b = \frac{y}{\surd(1+pp)}$, ſeu $\surd(1+pp) = \frac{y}{ny+b}$; unde fit $p = \frac{\surd(y^2-(ny+b)^2)}{ny+b} = \frac{dy}{dx}$; ac $dx = \ldots$ $\frac{(ny+b)\,dy}{\surd((1-n^2)y^2 - 2bny - bb)}$. Ex qua patet, ſi ſit $b = 0$, tum curvam eſſe abituram in lineam rectam puncta a & z jungentem. Deinde ſi ſit $n = 0$, ob $dx = \frac{b\,dy}{\surd(yy-bb)}$, curva erit Catenaria concavitatem axi AZ obvertens. Quod ſi autem ſit $n = -1$, fiet $dx = \frac{(b-y)\,dy}{\surd(2by-bb)}$; ex qua integrando oritur $x = c + \frac{2b-y}{3b}\surd(2by-bb)$; quæ eſt pro curva algebraïca, & in rationalibus præbet $9b(x-c)^2 = (2b-y)^2(2y-b)$. Eſt ideo linea tertii ordinis & pertinet ad ſpeciem 68 NEWTONI.

EXEMPLUM VI.

46. *Inter omnes curvas* az *ejuſdem longitudinis; definire eam quæ circa axem* AZ *rotata producat maximum ſolidum.*

Inter

Inter omnes igitur curvas proprietate communi $\int dx\,\surd(1+pp)$ gaudentes, ea quæritur in qua ſit $\int yydx$ maximum. Quoniam ergo formulæ $\int dx\ \surd(1+pp)$ valor differentialis eſt $= -n\nu.\, d.\, \frac{p}{\surd(1+pp)}$; formulæ vero $\int yydx$ valor differentialis eſt $= 2\, n\nu.\, y\, dx$; habebitur pro curva quæſita iſta æquatio $2\, y\, dx = \pm\, bb\, d.\, \frac{p}{\surd(1+pp)}$, quæ, multiplicata per p & integrata, dabit $yy + bc = \pm \frac{bb}{\surd(1+pp)}$; ſeu $\surd(1+pp) = \frac{\pm bb}{yy+bc}$; hincque $p = \frac{\surd(b^4 - (yy+bc)^2)}{yy+bc} = \frac{dy}{dx}$; ex qua fit $x = \int \frac{(yy+bc)\,dy}{\surd(b^4 - (yy+bc)^2)}$. Hæc curva hanc habet proprietatem ut ejus radius oſculi, qui generaliter eſt $= dx : d.\frac{p}{\surd(1+pp)}$, fiat $= \frac{bb}{2y}$; hoc eſt, proportionalis eſt applicatæ y inverſæ; unde patet curvam quæſitam eſſe Elaſticam. Non ſolum autem per conſtantes b & c arbitrarias effici poteſt, ut curva per datos terminos a & z tranſeat, ſed etiam ut ejus arcus intra hos terminos interceptus fiat datæ magnitudinis. Si fit $c = 0$, prodit Elaſtica rectangula. Cæterum nullo caſu conſtructio per quadraturam vel Circuli vel Hyperbolæ abſolvi poteſt; niſi ſint vel b & c infinita, quo quidem caſu linea a z prodit recta, vel $b = c$. Hoc enim caſu, habebitur $x = \int \frac{(yy+bb)\,dy}{y\surd - (2bb - yy)}$, ſeu, ſumto bb negativo, erit $x = \int \frac{(yy-bb)\,dy}{y\surd(2bb-yy)} = \int - \surd(2bb-yy) - bb\int \frac{dy}{y\surd(2bb-yy)}$; & integratione per logarithmos abſoluta, fiet $x = \int - \surd(2bb - yy) + b\, l \frac{b+\surd(2bb-yy)}{y}$. Ipſa vero curvæ longitudo, quæ generaliter eſt $= \int \frac{bb\,dy}{\surd(b^4 - (yy+bc)^2)}$, erit hoc caſu $= g - b\, l\, \frac{b+\surd(2bb-yy)}{y}$.

 EXEM-

Exemplum VII.

47. *Invenire curvam, qua, inter omnes alias ejuſdem longitudinis, circa axem* AZ *rotata, producat ſolidum cujus ſuperficies ſit vel maxima vel minima.*

Quoniam proprietas communis eſt $= \int dx \surd(1+pp)$, cujus valor differentialis eſt $- nv.\, d. \frac{p}{\surd(1+pp)}$; maximi minimive formulæ autem $\int y\,dx \surd(1+pp)$ valor differentialis eſt $= nv.\,(dx\surd(1+pp) - d.\frac{yp}{\surd(1+pp)})$; habebitur pro curva quæſita iſta æquatio $b\,d.\frac{p}{\surd(1+pp)} = dx\surd(1+pp) - d.\frac{yp}{\surd(1+pp)}$, quæ per p multiplicata & integrata præbet $c - \frac{b}{\surd(1+pp)} = \frac{y}{\surd(1+pp)}$, ſeu $c = \frac{b+y}{\surd(1+pp)}$. Hinc fiet $\surd(1+pp) = \frac{b+y}{c}$, & $p = \frac{\surd((b+y)^2 - cc)}{c} = \frac{dy}{dx}$: ex hacque $dx = \frac{c\,dy}{\surd((b+y)^2 - cc)}$, quæ eſt æquatio generalis pro Catenaria, & ſatisfacit, dummodo axis reſpectu catenæ ſuſpenſæ ſitum teneat horizontalem. Fieri igitur poteſt, ut curva vel convexitatem vel concavitatem axi AZ obvertat, priori caſu ſuperficies ſolidi fiet minima, poſteriori maxima.

Exemplum VIII.

Fig. 17. 48. *Inter omnes curvas per puncta* A & C *tranſeuntes, quæ omnes æquales areas* ABC *comprehendant; definire eam quæ in fluido ſecundum directionem axis* BA *mota minimam patiatur reſiſtentiam.*

Poſitis abſciſſa $AP = x$, applicata $PM = y$, proprietas communis eſt $\int y\,dx$, ejuſque valor differentialis $= nv.\,dx$. Reſiſtentia autem totalis, quam figura in directione AB ſentit, eſt ut

ut $\int \frac{p^3\,dx}{1+pp}$, cujus valor differentialis $-\,nv.\,d.\,\frac{3pp+p^4}{(1+pp)^2}$. Ex his emergit pro curva iſta æquatio $dx = bd.\,\frac{3pp+p^4}{(1+pp)^2}$; quæ integrata dat $x = c + \frac{bpp(3+pp)}{(1+pp)^2}$. Æquatio autem differentialis per p multiplicata, abit in hanc $dy = bpd.\,\frac{3pp+p^4}{(1+pp)^2}$, quæ in hanc formam $dy = bpd\,\frac{3pp+p^4}{(1+pp)^2} + bdp.\,\frac{3pp+p^4}{(1+pp)^2} - bdp.\,\frac{3pp+p^4}{(1+pp)^4}$ tranſmutata, habet integrale $y = f + \frac{bp^3(3+pp)}{(1+pp)^2} - \frac{bp^3}{1+pp}$ ſeu $y = f + \frac{2bp^3}{(1+pp)^2}$; cum igitur ſit $x = c + \frac{bpp(3+pp)}{(1+pp)^2}$, curva erit algebraïca. Efficiendum eſt autem, ut, quo caſu fit $x = 0$ [quod fieri nequit, niſi vel b vel c capiatur negativum] ſimul y evaneſcat. Quo autem curva cognoſcatur, ponatur $x - c = t$ & $y - f = u$, erit $t = \frac{bpp(3+pp)}{(1+pp)^2}$, & $u = \frac{2bp^3}{(1+pp)^2}$; unde fit $t + u\sqrt{3} = \frac{b(p^4 + 2p^3\sqrt{3} + 3pp)}{(1+pp)^2}$ atque $t - u\sqrt{3} = \frac{b(p^4 - 2p^3\sqrt{3} + 3pp)}{(1+pp)^2}$. Extrahendis igitur radicibus quadratis habebitur $\sqrt{\frac{t+u\sqrt{3}}{b}} = \frac{pp+p\sqrt{3}}{b}$, & $\sqrt{\frac{t-u\sqrt{3}}{b}} = \frac{pp-p\sqrt{3}}{1+pp}$; hincque $\sqrt{\frac{t+u\sqrt{3}}{b}} + \sqrt{\frac{t-u\sqrt{3}}{b}} = \frac{2pp}{1+pp}$, & $\sqrt{\frac{t+u\sqrt{3}}{b}} - \sqrt{\frac{t-u\sqrt{3}}{b}} = \frac{2p\sqrt{3}}{1+pp}$: At eſt $\frac{t}{b} = \frac{3}{2}.\,\frac{2pp}{1+pp} - \frac{1}{2}.\,\frac{4p^4}{(1+pp)^2} = \frac{3}{2}\sqrt{\frac{t+u\sqrt{3}}{b}} + \frac{3}{2}\sqrt{\frac{t-u\sqrt{3}}{b}} - \frac{1}{2}\left(\frac{2t}{b} + 2\sqrt{\frac{tt-3uu}{bb}}\right)$. Ergo $\frac{4t}{b} = 3\sqrt{\frac{t+u\sqrt{3}}{b}} + 3\sqrt{\frac{t-u\sqrt{3}}{b}} - \frac{2\sqrt{(tt-3uu)}}{b}$; quæ rationalis facta præbet æquationem hanc quarti ordinis

$4t^4 + 8ttuu + 4u^4 = 4bt^3 + 36btuu - 27b^2uu$, ſeu $4(tt+uu)^2 = 4bt^3 + 36btu^2 - 27b^2u^2$.

Ad curvam autem per infinita puncta conſtruendam, expedit adhibere has formulas, $t = \frac{b(p^4+3pp)}{(1+pp)^2}$ & $u = \frac{2bp^3}{(1+pp)^2}$. Primum autem patet curvam habere diametrum in poſitione abſciſſarum t ſitam, duobuſque locis fieri $u = 0$, nempe caſu $p = 0$, quo ſimul fit $t = 0$, & caſu $p = \infty$, quo fit $t = b$. Quod ſi ponatur $b = 4c$, atque $t = 3c + r$, orietur iſta æquatio $(rr+uu)^2 + 8c(r^3 - 3ru^2) + 18cc(r^2+u^2) - 27c^4 = 0$ quæ cum ſit functio ipſarum $rr+uu$ & $r^3 - 3ruu$, declarat curvam hanc habere tres diametros ſeſe in initio abſciſſarum harum r decuſſantes. Curva ergo quæſita triangulo æquilatero ABC ita erit inſcriptibilis, ut conſtet ex tribus ramis ADB, AEC & BFC inter ſe ſimilibus & æqualibus, qui in punctis A, B, & C cuſpides forment acutiſſimos. Ejus igitur diametri erunt tres rectæ AI, BH & CG, ſeſe in centro trianguli O decuſſantibus. Erit autem $AO = 3c$, $OF = c$, & $OI = \frac{3}{2}c$, ita ut ſit $AI = \frac{9}{2}c$ & $FI = \frac{1}{2}c = \frac{1}{2}OF$. Hujus jam curvæ quæcunque portio abc rectis ab & bc parallelis ipſis AI & BI & arcu curvæ ac comprehenſa, ita erit comparata, ut arcus ac inter omnes alios puncta a & c jungentes, & æqualem aream abc continentes, in fluido ſecundum directionem ba mota minimam patiatur reſiſtentiam. Porro autem hæc curva erit rectificabilis, reperiturque arcus $ADB = \frac{16}{3}c$; ex quo erit $ADB : AI = \frac{16}{3} : \frac{9}{2} = 32 : 27$, atque $ADB : AB = 32 : 18\sqrt{3} = 16 : 9\sqrt{3}$.

Exemplum IX.

Fig. 19. 49. *Inter omnes curvas* AM *æquales areas* APM *includentes; invenire eam, quæ ſit ita comparata, ut, ſi perpetuo a centro circuli oſculantis* O *ad applicatam* MP *productam ducatur perpendicularis* ON; *curva a punctis* N *formata minimam comprehendat aream,* APN.

Poſitis abſciſſa $AP = x$, & applicata $PM = y$; erit area APM

$APM = \int y\,dx$, quæ eſt proprietas communis, ejuſque valor differentialis $= nv.\,dx$. Deinde, cum ſit radius oſculi MO $= -\frac{(1+pp)^{3:2}}{q}$, fiet $MN = -\frac{(1+pp)}{q}$ & $PN = -\frac{(1+pp)}{q} - y$; ex quo area APN erit $= -\int y\,dx - \int \frac{(1+pp)}{q}\,dx$; quæ debet eſſe minima, cujus valor differentialis eſt $= nv.\,(-dx + d.\,\frac{2p}{q} + \frac{1}{dx}\,dd.\,\frac{(1+pp)}{qq})$; unde iſta naſcitur æquatio $n\,dx^2 = dx\,d.\,\frac{2p}{q} + dd.\,\frac{(1+pp)}{qq}$; quæ integrata dat $nx\,dx = \frac{2p\,dx}{q} + d.\,\frac{1+pp}{qq} + b\,dx$. Illa vero eadem æquatio, per p multiplicata, dat $n\,dx\,dy = dy\,d.\,\frac{2p}{q} + p\,dd.\,\frac{1+pp}{qq}$; cujus integrale eſt $ny\,dx = c\,dx - \frac{2\,dx}{q} + p\,d.\,\frac{1+pp}{qq}$. His æquationibus conjungendis, oritur $nx\,dy - ny\,dx = b\,dy - c\,dx + \frac{2p\,dy}{q} + \frac{2\,dx}{q} = b\,dy - c\,dx + \frac{2\,dx^2 + 2\,dy^2}{dp}$. Ponatur $nx - b = nt$, & $ny - c = nu$; erit $dy = du$, & $dx = dt$, atque $n\,dp = \frac{2\,dt^2 + 2\,du^2}{t\,du - u\,dt} = \frac{n\,ddu}{dt}$, ſeu $2\,dt^3 + 2\,dt\,du^2 = nt\,du\,ddu - nu\,dt\,ddu$, poſito dt conſtante. Sit $u = st$, erit $du = s\,dt + t\,ds$, & $ddu = t\,dds + 2\,dt\,ds$; hiſque ſubſtitutis prodibit iſta æquatio: $2(1+ss)\,dt^3 + 4st\,dt^2\,ds + 2(1-n)\,tt\,dt\,ds^2 = nt^3\,ds\,dds$. Ponatur $t = e^{\int r\,ds}$, erit $dt = e^{\int r\,ds}\,r\,ds$, & $ddt = 0 = e^{\int r\,ds}(r\,dds + dr\,ds + rr\,ds^2)$; unde fit $dds = -\frac{dr\,ds}{r} - r\,ds^2$; ex quibus tandem emergit $2(1+ss)\,r^3\,ds + 4sr^2\,ds + 2(1-n)\,r\,ds = -\frac{n\,dr}{r} - nr\,ds$, ſeu $\frac{n\,dr}{r} + (2-n)\,r\,ds +$

$4sr^2 ds + 2r^3 ds + 2r^3 s^2 ds = 0$. Sit $s = v - \frac{1}{r}$, fiet $dr + rrdv = \frac{ndv}{2(1+vv)}$; quæ æquatio integrationem admittit, quoties eſt $n = 2i(i-1)$ denotante i numerum integrum quemcunque: ut ſi ſit $n = 4$, fiet $r = \frac{2v}{1+vv} + \frac{1}{(1+vv)^2 \int \frac{dv}{(1+vv)^2}}$; ex qua retrogrediendo conſtructio abſolvi poterit.

EXEMPLUM X.

50. *Inter omnes curvas, in quibus* $\int xTdx$ *eundem obtinet valorem; invenire eam in qua ſit* $\int yTdx$ *maximum vel minimum, exiſtente* T *functione quacunque ipſius* p, *ita ut ſit* $dT = Pdp$.

Ad formulæ $\int xTdx$ valorem differentialem inveniendum; notandum eſt eſſe $d.xT = Tdx + xPdp$, ex quo illius valor differentialis erit $= -$ nν. $d.xP$. Ex altera autem formula $\int yTdx$ habetur $d.yT = Tdy + yPdp$, unde ejus valor differentialis erit nν. $(Tdx - d.yP)$. Quare, pro curva quæſita orietur iſta æquatio $nd.xP = Tdx - d.yP$. Ergo $\int Tdx = nxP + yP + b$. Porro ſi illa æquatio per p multiplicetur, habebitur $npd.xP = Tdy - pd.yP = d.yT - yP.dp - pd.yP = d.yT - d.yPp$. At eſt $pd.xP = Ppdx + pxdP + xPdp + Tdx - d.xT = d.xPp + Tdx - dxT$. Quamobrem orietur $d.yT - d.yPp = nd.xPp + nTdx - nd.xT$; hincque $\int nTdx = yT - yPp - nxPp + nxT + c$. Quia vero, ex ſuperiori integratione habemus $\int nTdx = nnxP + nyP + nb$, erit, eliminando $\int nTdx$, iſta æquatio $nnxP + nyP + nb = yT - yPp - nxPp + nxT + c$, ſeu $y = \frac{nx(nP + Pp - T) + c}{-nP - Pp + T}$, vel $y = -nx + \frac{c}{T - nP - Pp}$.

Ergo

Ergo prodit tandem $x = c\int \frac{dP}{(T - nP - Pp)^2}$, atque $y = c\int \frac{p\,dP}{(T - nP - Pp)^2} = \frac{c}{T - nP - Pp} - \ldots \ldots$ $nc\int \frac{dP}{(T - nP - Pp)^2}$.

Exemplum XI.

51. *Invenire curvam, quæ, inter omnes alias intra eoſdem terminos contentas, & eundem formulæ* $\int x\,dx\sqrt{(1+pp)}$ *valorem continentes, habeat* $\int y\,dx\sqrt{(1+pp)}$ *maximum vel minimum.*

Exemplum hoc eſt Caſus præcedentis, atque ex illo manat, ponendo $T = \sqrt{(1+pp)}$, ex quo erit $P = \frac{p}{\sqrt{(1+pp)}}$, & $dP = \frac{dp}{(1+pp)^{3:2}}$. Porro vero erit $T - nP - Pp = \frac{1 - np}{\sqrt{(1+pp)}}$. Ex his jam ſurrogatis, prodibit $x = c\int \frac{dp}{(1 - np)^2\sqrt{(1+pp)}}$ & $y = \frac{c\sqrt{(1+pp)}}{1 - np} - nx$. Integratione autem per logarithmos inſtituta fiet . . . .
$x = \frac{nc(p + \sqrt{(1+pp)}) - c}{(1+nn)(1-np)} + \frac{c}{(1+nn)^{3:2}} \times$
$l\,\frac{n + (1+\sqrt{(1+nn)})(p+\sqrt{(1+pp)})}{n + (1-\sqrt{(1+nn)})(p+\sqrt{(1+pp)})} + b$, &
$y = \frac{nc + c(\sqrt{(1+pp)} - nnp)}{(1+nn)(1-np)} - \frac{nc}{(1+nn)^{3:2}} \times$
$l\,\frac{n + (1+\sqrt{(1+nn)})(p+\sqrt{(1+pp)})}{n + (1-\sqrt{(1+nn)})(p+\sqrt{(1+pp)})} - nb$; ex quibus valoribus curva conſtrui poterit per logarithmos. Generaliter autem, quamcunque T functionem ipſius p denotet, conſtructio ſemper per quadraturas abſolvi poteſt. Cæterum hoc Exemplum ſine ſubſidio præcedentis multo difficilius ſolutu fuiſ-

Digitized by Google

ſet; non tam facile enim perſpicere licuiſſet, quomodo æquatio inventa integrabilis redderetur quam in caſu generali.

PROPOSITIO V. PROBLEMA.

52. *Inter omes curvas ad eandem abſciſſam* $= a$ *relatas, quæ eundem formula* $\Pi = \int [Z] dx$ *valorem recipiunt; invenire eam, in qua ſit* $\int Z dx$ *maximum vel minimum; exiſtente* Z *functione ſimul ipſius* Π, *ita ut ſit* $dZ = L d\Pi + M dx + N dy + P dp + Q dq +$ &c. *atque* $d[Z] = [M] dx + [N] dy + [P] dp + [Q] dq +$ &c.

SOLUTIO.

Quoniam eſt $d[Z] = [M] dx + [N] dy + [P] dp + [Q] dq +$ &c. erit formulæ $\int [Z] dx$, quæ hic quantitatem omnibus curvis communem repræſentat, valor differentialis $= n \nu . dx ([N] - \frac{d[P]}{dx} + \frac{dd[Q]}{dx^2} -$ &c.$)$, qui ex Caſu primo §. 7 Cap. præced. ſequitur. At formula $\int Z dx$, maximum minimumve exprimens, quia Z involvit formulam integralem $\Pi = \int [Z] dx$, pertinet ad Caſum ſecundum loci citati: ejuſque adeo valor differentialis erit $= n \nu . dx (N + [N] V - \frac{d(P + [P] V)}{dx} + \frac{dd(Q + [Q] V)}{dx^2} -$ &c.$)$, denotante $V = H - \int L dx$, ubi H eſt quantitas determinata, quæ oritur ſi in integrali $\int L dx$ ponatur $x = a$. Atque, ob hanc ipſam quantitatem H, iſte valor differentialis a præſcripta longitudine abſciſſæ $x = a$ pendet. Ex his igitur duobus valoribus differentialibus ambarum formularum propoſitarum, quarum altera proprietatem communem, altera maximum minimumve exponit, ſecundum regulam datam, naſcitur æquatio pro curva ſequens: $0 = \alpha [N] - \frac{\alpha d[P]}{dx} + \frac{\alpha dd[Q]}{dx^2} -$ &c. $+ N + [N] V - \frac{d(P + [P] V)}{dx} + \frac{dd(Q + [Q] V)}{dx^2} -$ &c. quæ, ob $V =$

$H -$

$H - \int L dx$, tranſit in hanc $o = N + (\alpha + H - \int L dx [N] - \frac{d(P + (\alpha + H - \int L dx)[P])}{dx} + \frac{dd(Q + (\alpha + H - \int L dx)[Q])}{dx^2}$ — &c.

Cum jam α ſit quantitas conſtans arbitraria; etiamſi H ſit quantitas conſtans determinata, tamen $\alpha + H$ fiet quantitas arbitraria: ideoque non amplius a definita abſciſſæ longitudine a pendet. Quare ſi, loco $\alpha + H$, ſcribamus C, habebimus pro curva quæſita hanc æquationem:

$$o = N + (C - \int L dx)[N] - \frac{d(P + (C - \int L dx)[P])}{dx} + \frac{dd(Q + (C - \int L dx)[Q])}{dx^2}$$

— &c. quæ ergo pro quacunque abſciſſa exhibet Curvam, quæ, inter omnes alias eundem formulæ $\int [Z] dx$ valorem recipientes, continebit formulæ $\int Z dx$ maximum minimumve valorem. Q. E. I.

COROLL. I.

53. Si igitur proprietas communis fuerit ea ipſa formula integralis, quæ in maximi minimive formula implicatur; tum conſideratio determinatæ abſciſſæ magnitudinis ex calculo egreditur, & Curva inventa pro quavis abſciſſa quæſito ſatisfaciet.

COROLL. II.

54. In hac æquatione inventa, duæ adhuc inerunt formulæ integrales; primo nempe formula $\int L dx$, ac deinde formula $\pi = \int [Z] dx$, quæ cum ea in Z contineatur, inerit in quantitatibus L, M, N, P &c.

COROLL. III.

55. Si igitur hæc integralia per differentiationem tollere lubeat; pervenietur ad differentialia binis gradibus altiora, ſimulque exibit conſtans arbitraria C. Interim tamen numerus conſ-

tantium arbitrariarum unitate minor erit quam gradus iſte differentialium; eo quod integrale $\Pi = \int [Z]\, dx$ definitum obtinere debet valorem, eum ipſum ſcilicet, quem in maximi minimive formula $\int Z\, dx$ habet.

Coroll. IV.

56. Hinc igitur in æquatione inventa, ob conſtantem arbitrariam C, poteſtate una plures inerunt conſtantes, quam differentialium gradus indicat. Quarum una eo determinabitur, ut valor formulæ communis $\Pi = \int [Z]\, dx$ fiat pro curva inventa datæ magnitudinis; reliquæ vero per data puncta vel tangentium poſitionem datam determinabuntur.

Coroll. V.

57. Si Z fuerit functio cum quantitatum x, y, p, q, &c. tum arcus curvæ s: atque inter omnes Curvas iſoperimetras quæratur ea, in qua ſit $\int Z dx$ maximum vel minimum; tum fiet $\Pi = s = \int [Z]\, dx$ & $[Z] = \surd(1+pp)$, ita ut ſit $[M] = 0$, $[N] = 0$, & $[P] = \frac{p}{\surd(1+pp)}$.

Coroll. VI.

58. Hoc igitur caſu, ſi fuerit $dZ = L\,ds + M\,dx + N\,dy + P\,dp + Q\,dq +$ &c. habebitur pro Curva quæ, inter omnes iſoperimetras, habeat $\int Zdx$ maximum vel minimum, iſta æquatio:

$$0 = N - \frac{1}{dx}\, d\left(P + \frac{(C - \int L\,dx)\,p}{\surd(1+pp)}\right) + \frac{ddQ}{dx^2} - \text{\&c.}$$

ſeu

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} = \frac{(C - \int L\,dx)\,dp}{dx\,(1+pp)^{3:2}} - \frac{Lp}{\surd(1+pp)}$$

ſive

$$\frac{Lp}{\surd(1+pp)} + N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \text{\&c.} = \frac{(C - \int Ldx)\,dp}{dx\,(1+pp)^{3:2}}$$

Co-

COROLL. VII.

59. Cum ſit C quantitas arbitraria; in genere notari convenit, quod ſi pro C accipiatur ille formulæ $\int L\,dx$ valor, quem inducit ſi ponatur $x = a$, tum prodituram eſſe curvam, quæ inter omnes omnino curvas eidem abſciſſæ $x = a$ reſpondentes, habeat valorem formulæ $\int Z\,dx$ maximum vel minimum.

SCHOLION I.

60. Caſus Coroll. 6, quia is ab Auctoribus potiſſimum tractari eſt ſolitus, peculiarem evolutionem meretur, ut ejus ope Problemata quæ forte occurrere queant, facilius & expeditius reſolvi poſſint. Inter omnes igitur Curvas iſoperimetras, ſeu quæ eandem habeant longitudinem $s = \int dx\sqrt{(1+pp)}$, quæratur ea, in qua ſit $\int Z\,dx$ maximum vel minimum, exiſtente Z functione cum quantitatum definitarum x, y, p, q &c. tum arcus curvæ s; ita ut ſit $dZ = L\,ds + M\,dx + N\,dp + P\,dp +$ &c. Pro curva hac proprietate gaudente jam inventa eſt hæc æquatio: $\frac{1}{dx}\,d.\frac{(C - \int L\,dx)p}{\sqrt{(1+pp)}} = N - \frac{dP}{dx} + \frac{ddQ}{dx^2} -$ &c. quæ quidem, in hoc latiſſimo ſenſu nec integrari nec ad ſimpliciorem formam ſe reduci patitur. At caſus notaſſe juvabit, quibus eam integrare licebit. Ac primo quidem ſi ſit $N = 0$ ſponte prodit iſta pro curva æquatio:

$$A + \frac{(C - \int L\,dx)p}{\sqrt{(1+pp)}} = -P + \frac{dQ}{dx} - \text{\&c.}$$

jam ſemel integrata. Secundo ponamus eſſe $M = 0$; atque æquatio per $p\,dx = dy$ multiplicata abibit in hanc

$$p\,d.\frac{(C - \int L\,dx)p}{\sqrt{(1+pp)}} = N\,dy - P\,dp + \frac{p\,ddQ}{dx} - \text{\&c.}$$

ad quam ſi addatur $L\,ds = L\,dx\sqrt{(1+pp)} = dZ - N\,dy - P\,dp - Q\,dq$ &c.; integratione inſtituta prodibit

$$\int\left(L\,dx\sqrt{(1+pp)} + \frac{p\,d.(C - \int \mathit{E}\,dx)p}{\sqrt{(1+pp)}}\right) = Z - Pp - Qq + \frac{p\,dQ}{dx}\ \text{\&c.}$$

Prius

Digitized by Google

Prius vero membrum ſi evolvatur, tranſit in $\int(L\,dx\,\surd(1+pp) + \frac{(C-\int L\,dx)p\,dp}{(1+pp)^{3:2}} - \frac{Lpp\,dx}{\surd(1+pp)}) = \int(\frac{L\,dx}{\surd(1+pp)} + \frac{(C-\int L\,dx)p\,dp}{(1+pp)^{3:2}})$, cujus integrale eſt $-\frac{C-\int L\,dx}{\surd(1+pp)}$. Quare, caſu quo $M=0$, habebitur iſta æquatio $\frac{C-\int L\,dx}{\surd(1+pp)} = A - z + Pp + Qq - \frac{p\,dQ}{dx}$. Sin autem tertio fuerit tam $M=0$ quam $N=0$, habebitur primum, ob $N=0$, hæc æquatio: $A + \frac{(C-\int L\,dx)p}{\surd(1+pp)} = -P + \frac{dQ}{dx}$; quæ, multiplicata per $dp = q\,dx$, abit in hanc $A\,dp + \frac{(C-\int L\,dx)p\,dp}{\surd(1+pp)} = -P\,dp + Q\,dq$. Cum autem ſit $dZ = L\,dx\,\surd(1+pp) + P\,dp + Q\,dq$, habebitur $dz + A\,dp - L\,dx\,\surd(1+pp) + \frac{(C-\int L\,dx)p\,dp}{\surd(1+pp)} = q\,dQ + Q\,dq$; quæ integrata dabit, $Z + B + Ap + (C-\int L\,dx)\surd(1+pp) = Qq$, ſeu $C - \int L\,dx = \frac{Qq - B - Ap - Z}{\surd(1+pp)}$. At ex priore æquatione eſt $C - \int L\,dx = -\frac{A\surd(1+pp)}{p} - \frac{P\surd(1+pp)}{p} + \frac{dQ\surd(1+pp)}{p\,dx}$; ex quibus conjungendis elicitur: $A\,dx - B\,dy = z\,dy - P\,dx - Pp\,dy + dQ + pp\,dQ - Qp\,dp$, in qua non amplius ineſt formula integralis $\int L\,dx$. Uſum igitur horum caſuum in Exemplis monſtrabimus.

Exemplum I.

61. *Inter omnes curvas iſoperimetras; definire eam, in qua ſit* $\int s^n\,dx$ *maximum vel minimum, denotante* s *arcum curvæ abſciſſæ* x *reſpondentem.*

Quo-

Digitized by Google

Quoniam proprietas communis longitudinem arcus $s = \int dx \sqrt{(1 + pp)}$ reſpicit, atque in maximi minimive formula $\int s^n dx$ ineſt ipſe arcus, ſolutio pertinebit ad Caſum in Scholio pertractatum. Comparata ergo formula $\int s^n dx$ cum generali $\int Z dx$, fiet $Z = s^n$ & $dZ = n s^{n-1} ds$; hincque $L = n s^{n-1}$, $M = 0$, $N = 0$, $P = 0$ &c. Quare ex Scholii Caſu ultimo, quo poſueramus $M = 0$ & $N = 0$, habebitur iſta æquatio $A dx - B dy = Z dy = s^n dy$, ex qua oritur $A dx = dy (B + s^n)$ & $A^2 dx^2 + A^2 dy^2 = A^2 ds^2 = dy^2 (A^2 + (B + s^n)^2)$ ideoque $dy = \frac{A ds}{\sqrt{(A^2 + (B + s^n)^2)}}$ atque $dx = \frac{(B + s^n) ds}{\sqrt{(A^2 + (B + s^n)^2)}}$; unde Curvæ conſtructio perfici poterit. Vel poſito $dy = p dx$, erit $s^n = \frac{A - Bp}{p}$, atque $s = \sqrt[n]{\frac{A - Bp}{p}}$: ex quo fiet $ds = dx \sqrt{(1 + pp)} = - \frac{A dp (A - Bp)^{(1-n):n}}{n p^{(1+n):n}}$. Atque hinc per p coordinatæ curvæ x & y ita determinabuntur, ut ſit $x = - \frac{A}{n} \int \frac{dp (A - Bp)^{(1-n):n}}{p^{(1+n):n} \sqrt{(1 + pp)}}$ & $y = - \frac{A}{n} \int \frac{dp (A - Bp)^{(1-n):n}}{p^{1:n} \sqrt{(1 + pp)}}$. Videntur hic quidem quatuor conſtantes, duæ ſcilicet novæ, præter A & B, ingredi, ob duplicem integrationem y & x. At cum poſito $x = 0$, ſimul arcus curvæ $s = \sqrt[n]{\frac{A - Bp}{p}}$ evaneſcere debeat; hinc viciſſim conſtans in integratione ipſius x orta definietur. Nimirum ſi n fuerit numerus affirmativus, arcus

Digitized by Google

cus s evaneſcit, poſito $p = \frac{A}{B}$; ex quo valor ipſius x ita determinari debet, ut poſito $p = \frac{A}{B}$ fiat $= 0$.

Quod ſi ponatur $n = 1$; habebitur ex priore conſtructione, ſtatim $dx = \frac{(B+s)\,ds}{\surd(A^2 + (B+s)^2)}$; ideoque $x = \surd(A^2 + B^2 + 2Bs + ss) - \surd(A^2 + B^2)$, ſeu poſito $B = b$, & $\surd(A^2 + B^2) = c$, erit $x + c = \surd(c^2 + 2bs + ss)$. Ex poſteriore autem conſtruendi modo, oritur $x = - A\int\frac{dp}{pp\surd(1+pp)} = \frac{A\surd(1+pp)}{p} + b$, ſeu $(x - b)p = c\surd(1+pp)$; hincque $p = \frac{c}{\surd((x-b)^2 - c^2)} = \frac{dy}{dx}$. Quare cum ſit $y = \int\frac{c\,dx}{\surd((x-b)^2 - c^2)}$; curva ſatisfaciens erit Catenaria.

EXEMPLUM II.

62. *Inter omnes curvas ejuſdem longitudinis, eam determinare, in qua ſit* $\int S\,dx$ *maximum vel minimum, exiſtente* S *functione quacunque arcus* s.

Quia proprietas communis arcu $s = \int dx\surd(1+pp)$ continetur; ſolutio ex Scholio peti poterit. Scilicet cum ſit $Z = S =$ functioni ipſius s, erit $L\,ds = dS$, & $M = N = P = Q$ &c. $= 0$. Quare, per tertium Scholii Caſum, habebitur pro curva quæſita iſta æquatio $A\,dx - B\,dy = S\,dy$, & $A\,dx = dy(B+S)$. Hinc ergo erit $A^2dx^2 + A^2dy^2 = A^2ds^2 = dy^2(A^2 + (B+S)^2)$ & $y = \int\frac{A\,ds}{\surd(A^2 + (B+S)^2)}$; erit autem abſciſſa $x = \int\frac{(B+S)\,ds}{\surd(A^2 + (B+S)^2)}$; unde curvæ conſtructio abſolvi poterit.

Ponamus eſſe $S = e^s$; poſitoque $dy = p\,dx$, erit $\frac{A - Bp}{p} =$

$= e^s$, & $e^s ds = \frac{-A dp}{pp} = \frac{(A - Bp) dx \sqrt{(1+pp)}}{p}$, hincque $dx = \frac{-A dp}{(A - Bp) p \sqrt{(1+pp)}}$ & $dy = \frac{-A dp}{(A - Bp) \sqrt{(1+pp)}}$. Componendo vero fiet $dx - \frac{B dy}{A} = \frac{-dp}{p\sqrt{(1+pp)}}$, & integrando $Ax - By = A\, l\, \frac{1 + \sqrt{(1+pp)}}{p} + C$, ſeu $\frac{1 + \sqrt{(1+pp)}}{p} = e^{(Ax - By - C) : A}$. Cum autem, facto $s = 0$, evaneſcere debeat x, atque ob $\frac{A - Bp}{p} = e^s$, facto $s = 0$, fiat $p = \frac{A}{A+1}$, per integrationes efficiendum eſt, ut facto $p = \frac{A}{B+1}$ fiat $x = 0$.

EXEMPLUM III.

63. ***Inter omnes curvas ejuſdem longitudinis, determinare eam in qua ſit*** $\int y\,dx$ ***maximum vel minimum, denotante*** s ***arcum curvæ.***

Solutio hujus Quæſtionis iterum petenda eſt ex Scholio; erit namque $Z = sy$ & $dZ = y ds + s dy$, ex quo fit $L = y$, $M = 0$ & $N = s$, reliquæ litteræ P, Q, &c. evaneſcent. Cum igitur ſit $M = 0$, Caſus Scholii ſecundus hanc ſuppeditabit ſolutionem: $\frac{C - \int y dx}{\sqrt{(1+pp)}} = A - ys$; immediate vero prodit $s dx = d. \frac{(C - \int y dx) p}{\sqrt{(1+pp)}} = \frac{(C - \int y dx) dp}{(1+pp)^{3:2}} - \frac{yp dx}{\sqrt{(1+pp)}}$. Quare, cum ſit $C - \int y dx = A\sqrt{(1+pp)} - ys\sqrt{(1+pp)}$, erit $s dx = \frac{A dp - ys dp - y dy}{1+pp}$, ſeu $s dx + sp dy + ys dp + y dy = A dp$. Sin autem lubuerit arcum s eliminare, habebitur ex binis æquationibus, $s = \frac{A}{y} - \frac{(C - \int y dx)}{y\sqrt{(1+pp)}} = \frac{(C - \int y dx) dp}{dx (1+pp)^{3:2}}$

$-\frac{yp}{\surd(1+pp)}$; hincque $\frac{A\,dx}{y}+\frac{yp\,dx}{\surd(1+pp)}=(C-\int y\,dx)$ $\left(\frac{dp}{(1+pp)^{3:2}}+\frac{dx}{y\surd(1+pp)}\right)$. In utroque autem caſu difficile eſt ad æquationem ad curvam conſtruendum accommodatam pertingere.

EXEMPLUM IV.

64. *Inter omnes curvas eandem aream* $\Pi=\int y\,dx$ *continentes, definire eam, in qua ſit* $\int\frac{dx\surd(1+pp)}{\Pi}$ *maximum vel minimum.*

Si hanc Quæſtionem cum Solutione generali comparemus, habebimus $\int[Z]\,dx=\int y\,dx$; hincque $[Z]=y$, & $[N]=1$; reliquis litteris $[M]\,[P]\,[Q]$, &c. evaneſcentibus. Porro erit $Z=\frac{\surd(1+pp)}{\Pi}$, & $dZ=-\frac{d\Pi\surd(1+pp)}{\Pi^2}+\frac{p\,dp}{\Pi\surd(1+pp)}$, unde erit $L=-\frac{\surd(1+pp)}{\Pi^2}$, $M=0$, $N=0$, & $P=\frac{p}{\Pi\surd(1+pp)}$. Quocirca pro curva quæſita ſequens emerget æquatio:

$$0=C+\int\frac{dx\surd(1+pp)}{\Pi^2}-\frac{1}{dx}\,d.\,\frac{p}{\Pi\surd(1+pp)}.$$

Multiplicetur hæc æquatio per $dy=p\,dx$, erit $0=C\,dy+dy\int\frac{dx\surd(1+pp)}{\Pi^2}-p\,d.\,\frac{p}{\Pi\surd(1+pp)}$, quæ integrata dabit: $0=B+Cy+y\int\frac{dx\surd(1+pp)}{\Pi^2}-\int\frac{d\Pi\surd(1+pp)}{\Pi^2}-\frac{pp}{\Pi\surd(1+pp)}+\int\frac{p\,dp}{\Pi\surd(1+pp)}=B+Cy+y\int\frac{dx\surd(1+pp)}{\Pi^2}-\frac{pp}{\Pi\surd(1+pp)}+\frac{\surd(1+pp)}{\Pi}$. Hinc itaque iſtam obtinebimus æquationem $0=B+Cy+y\int\frac{dx\surd(1+pp)}{\Pi^2}+\frac{1}{\Pi\surd(1+pp)}$; a qua ſi prior per y multiplicata ſubtrahatur, erit

0 =

$$o = B + \frac{1}{\Pi\sqrt{(1+pp)}} + \frac{y}{dx}\, d.\, \frac{p}{\Pi\sqrt{(1+pp)}},\ \text{ſeu}$$

$$o = Bdx + \frac{dx}{\Pi\sqrt{(1+pp)}} + \frac{y\,dp}{\Pi(1+pp)^{3:2}} - \frac{y^2\,p\,dx}{\Pi^2\sqrt{(1+pp)}};$$

ex qua æquatione ſi denuo $\Pi = \int y\,dx$ exterminare velimus, prodiret æquatio differentialis tertii ordinis, ex qua multo minus quicquam ad Curvam cognoſcendam deduci poſſet.

SCHOLION II.

65. Quanquam, in hac Propoſitione poſuimus $[Z]$ eſſe functionem determinatam quantitatum x, y, p, q, &c. tamen Methodus ſolvendi patet, ſi hæc ipſa quantitas $[Z]$ fuerit functio indefinita formulas integrales in ſe complectens. Ponamus enim in formula $\Pi = \int [Z]\,dx$, quæ omnibus curvis debet eſſe communis, eſſe

$d[Z] = [L]d\pi + [M]dx + [N]dy + [P]dp + [Q]dq +$ &c.
exiſtente $\pi = \int [z]\,dx$, &

$d[z] = [m]\,dx + [n]\,dy + [p]\,dp + [q]\,dq +$ &c.

Maximum minimumve autem eſſe oportere formulam $\int Z\,dx$, exiſtente: $dZ = L\,d\Pi + M\,dx + N\,dy + P\,dp +$ &c. Jam formula $\int [Z]\,dx$ continetur in Caſu ſecundo §. 7 Cap. præc: inde ergo ſi capiatur integrale $\int [L]\,dx$ ejuſque valor reſpondens abſciſſæ $x = a$, ad quam ſolutio debet accommodari, ponatur $= [H]$, atque $[H] - \int [L]\,dx = [V]$; habebitur formulæ $\int [Z]\,dx$ valor differentialis $= n\nu.\,dx\,([N] + [n][V] - \frac{d([P] + [p][V])}{dx} + \frac{dd([Q] + [q][V])}{dx^2}$ — &c.). Deinde vero maximi minimive formula $\int Z\,dx$ continetur in Caſu tertio loci citati; ad ejuſque valorem differentialem inveniendum, ponatur formulæ $\int L\,dx$ valor abſciſſæ $x = a$ reſpondens $= H$, ac $H - \int L\,dx = V$. Jam capiatur integrale $\int [L]\,V\,dx = H\int [L]\,dx - \int [L]\,dx \int L\,dx$ ſitque, poſito $x = a$, valor formulæ $\int [L]\,dx \int L\,dx = K$, eodem autem caſu formulæ $\int [L]\,dx$ valor eſt $= [H]$, ex quo formulæ

$\int [L] V dx$, caſu $x = a$, valor erit $= H[H] - K$, & vocetur $H[H] - K - H\int [L] dx + \int [L] dx \int L dx = W$, ita ut ſit $W = H[V] - K + \int [L] dx \int L dx$, eritque formulæ propoſitæ $\int Z dx$ valor differentialis $= n\nu . dx \left(N + [N] V + [n] W - \frac{d. (P + [P] V + [p] W)}{dx} + \frac{dd (Q + [Q] V + [q] W)}{dx^2} - \text{\&c.}\right)$. Quod ſi jam ad hunc valorem differentialem addatur præcedens per quantitatem conſtantem arbitrariam α multiplicatus, ſummaque ponatur $= 0$, prodibit æquatio pro curva quæſita hæc:

$$0 = N + [N](\alpha + V) + [n](\alpha [V] + W) - \frac{1}{dx} \cdot d(P + [P](\alpha + V) + [p](\alpha [V] + W)) + \frac{1}{dx^2} \cdot dd(Q + [Q](\alpha + V) + [q](\alpha [V] + W)) - \text{\&c.}$$

Eſt vero hic $\alpha + V = \alpha + H - \int L dx$; unde ſi ponatur $\alpha + H = C$, erit C conſtans arbitraria, & $\alpha + V = C - \int L dx$, atque $\alpha [V] + W = C[H] - K - C\int [L] dx + \int [L] dx \int L dx$. Hoc igitur pacto, pervenietur ad curvam quæſitam, in cujus æquatione, quia ob $[H]$ & K adhuc ineſt conſtans data a, ea quæſito ſatisfaciet tantum pro propoſita abſciſſa $x = a$. Quod ſi autem formularum ambarum altera ad Caſum 4, altera ad Caſum 5 pertineat, tum iterum conſideratio datæ abſciſſæ a ex calculo egreditur, eademque curva pro omni abſciſſa ſatisfaciet, id quod unico ſequenti Exemplo declaraſſe ſufficiet.

EXEMPLUM V.

66. *Inter omnes curvas eidem abſciſſæ reſpondentes, quæ eundem formulæ* v *valorem recipiunt; invenire eam, in qua ſit* $\int \frac{dx\sqrt{(1+pp)}}{vv}$ *maximum vel minimum, exiſtente* $dv = g dx + W dx \sqrt{(1 + pp)}$ *& W functione quacunque ipſius* v.

Solutio hujus Quæſtionis exhibebit curvam ſuper qua corpus deſcen-

deſcendens a gravitate uniformi g deorſum, in directione abſciſſarum ſollicitatum in medio quocunque reſiſtente celerrime delabitur, inter omnes alias curvas ſuper quibus deſcendendo eandem acquirit celeritatem. Eſt enim $\sqrt{v}$ celeritas corporis in quocunque curvæ puncto, & W exprimit reſiſtentiam medii. Quod nunc primum ad proprietatem communem $v = \int dx(g + W\sqrt{(1+pp)})$; ponamus eſſe $dW = U\,dv$, atque hæc formula ad Caſum quartum pertinebit; erit namque $\Pi = v$, & $Z = g + W\sqrt{(1+pp)}$, ac $dZ = U\,dv\,\sqrt{(1+pp)} + \frac{Wp\,dp}{\sqrt{(1+pp)}}$; unde erit $L = U\sqrt{(1+pp)}$, $M = 0$, $N = 0$, & $P = \frac{Wp}{\sqrt{(1+pp)}}$. Sumatur ergo integrale $\int U\,dx\sqrt{(1+pp)}$, ſitque, caſu quo $x = a$ ponitur, $e^{\int U\,dx\sqrt{(1+pp)}} = H$, ac ponatur $V = He^{-\int U\,dx\sqrt{(1+pp)}}$. Ex his erit formulæ v valor differentialis $= \Pi\nu.\,dx\,(-\frac{1}{dx}\,d.\,\frac{WVp}{\sqrt{(1+pp)}}) = -\Pi\nu.\,d.\,\frac{WVp}{\sqrt{(1+pp)}}$. Porro maximi minimive formula $\int\frac{dx\sqrt{(1+pp)}}{\sqrt{v}}$ pertinebit ad Caſum quintum, eritque $Z = \frac{\sqrt{(1+pp)}}{\sqrt{v}}$, & $dZ = -\frac{dv\sqrt{(1+pp)}}{2v\sqrt{v}} + \frac{p\,dp}{\sqrt{v(1+pp)}}$; ideoque $\Pi = v$, & $L = -\frac{\sqrt{(1+pp)}}{2v\sqrt{v}}$; $M = 0$, $N = 0$, & $P = \frac{p}{\sqrt{v(1+pp)}}$. Deinde vero, ob $v = \int dx\,(g + W\sqrt{(1+pp)})$, erit $[Z] = g + W\sqrt{(1+pp)}$, & $d[Z] = U\,dv\,\sqrt{(1+pp)} + \frac{Wp\,dp}{\sqrt{(1+pp)}}$; unde $[L] = U\sqrt{(1+pp)}$, $[M] = 0$, $[N] = 0$, & $[P] = \frac{Wp}{\sqrt{(1+pp)}}$. Ponatur, ſi poſt integrationem fiat $x = a$, $-\int e^{\int U\,dx\sqrt{(1+pp)}}\frac{dx\sqrt{(1+pp)}}{2v\sqrt{v}} = K$,

ſitque

Digitized by Google

ſitque $e^{-\int U dx \surd(1+pp)} (K + \int e^{\int U dx \surd(1+pp)} \frac{dx \surd(1+pp)}{2v\surd v}) = T$: atque erit formulæ $\int \frac{dx \surd(1+pp)}{\surd v}$ valor differentialis $= nv.\,dx(\frac{1}{dx} d(\frac{p}{\surd v(1+pp)}) + \frac{WTp}{\surd(1+pp)}) = -nv.\,d(\frac{p}{\surd v(1+pp)} + \frac{WTp}{\surd(1+pp)})$. Ex his duobus valoribus differentialibus inventis naſcitur pro curva quæſita ſequens æquatio, $o = \alpha d.\frac{WVp}{\surd(1+pp)} + d(\frac{p}{\surd v(1+pp)} + \frac{WTp}{\surd(1+pp)})$, & integrando, $B = \frac{p}{\surd v(1+pp)} + \frac{Wp(\alpha V+T)}{\surd(1+pp)}$. At eſt $\alpha V + T = e^{-\int U dx \surd(1+pp)}(\alpha H + K + \int e^{\int U dx \surd(1+pp)}\frac{dx\surd(1+pp)}{2v\surd v})$. Quod ſi ergo ponatur $\alpha H + K = C$, erit C conſtans arbitraria, atque quantitas definita α omnino ex æquatione evaneſcet; ideoque curva quæſita deſideratam proprietatem pro quavis abſciſſa poſſidebit. Pro curva quæſita habebitur ergo iſta æquatio:

$e^{\int U dx \surd(1+pp)}(\frac{B\surd(1+pp)}{Wp} - \frac{1}{W\surd v}) = C + \int e^{\int U dx \surd(1+pp)}\frac{dx\surd(1+pp)}{2v\surd v}$, & differentiando $\frac{-Bdp}{Wp^2\surd(1+pp)} - \frac{BUdv\surd(1+pp)}{W^2 p} + \frac{Udv}{W^2\surd v} + \frac{dv}{2Wv\surd v} + \frac{BUdx(1+pp)}{Wp} - \frac{Udx\surd(1+pp)}{W\surd v} = \frac{dx\surd(1+pp)}{2v\surd v}$. Cum autem ſit $dv = gdx + Wdx\surd(1+pp)$, habebimus facta ſubſtitutione, hanc æquationem $\frac{Bdp}{Wp^2\surd(+pp)} = \frac{gdx}{2Wv\surd v} + \frac{gUdx}{W^2\surd v} - \frac{gBUdx\surd(1+pp)}{W^2 p}$, ſive iſtam $\frac{2BWdp}{\surd(1+pp)} = \frac{gWp^2 dx}{v\surd v} + \frac{2gUp^2 dx}{\surd v} - 2gBUpdx\surd(1+pp)$. Multiplicetur hæc æquatio per dv, & in primo termino loco dv ſcribatur $gdx +$ Wdx

$W dx \surd(1+pp)$, ac dW loco Udv; quo facto, habebitur iſta æquatio $\frac{2gBdp}{W\surd(1+pp)} + 2Bdp - \frac{gp^2 dv}{Wv\surd v} = \frac{2gppdW}{W^2\surd v} - \frac{2gBpdW\surd(1+pp)}{W^2}$; quæ diviſa per p^2 fit integralis; eritque æquatio integrata hæc, $2C - \frac{2B}{p} = \frac{2gB\surd(1+pp)}{Wp} - \frac{2g}{W\surd v}$, ſive $W = \frac{gB\surd v(1+pp) - gp}{Cp\surd v - B\surd v} = \frac{dv - gdx}{dx\surd(1+pp)}$. Unde naſcitur æquatio a reſiſtentia W libera hæc, $(Cp - B)dv = gCpdx + gBppdx - \frac{gpdx\surd(1+pp)}{\surd v}$. Cum autem W ſit functio ipſius v data, ope æquationis $W\surd v = \frac{gB\surd v(1+pp) - gp}{Cp - B}$, dabitur p per v; qui valor ſi in præcedente æquatione ſubſtituatur, dabitur dx per v & dv; hincque curva quæſita poterit conſtrui.

PROPOSITIO VI. PROBLEMA.

67. *Inter omnes curvas proprietate communi* A *præditas, determinare eam, in qua ſit functio quæcunque, cum ipſius illius expreſſionis* A, *tum alius cujuſcunque* B, *maximum vel minimum.*

S O L U T I O.

Sit dA valor differentialis expreſſionis A, atque dB valor differentialis expreſſionis B; habebit functionis illius ipſarum A & B, quam maximum minimumve eſſe oportet, valor differentialis hujuſmodi formam $\alpha dA + \beta dB$; in qua conſtantes α & β a ratione compoſitionis qua expreſſiones A & B in illa functione inter ſe permiſcentur pendent; ita ut valores obtineant determinatos ab abſciſſæ quantitate, cui ſolutionem accomodatam eſſe oportet pendentes. Quoniam vero expreſſionis A, quæ proprietatem communem complectitur, valor differentialis eſt

dA; hujus multiplum quodcunque γdA addatur ad valorem differentialem $\alpha dA + \beta dB$ expreſſionis, quæ maximum minimumve eſſe debet, ac ſumma $(\alpha + \gamma) dA + \beta dB$ nihilo æqualis poſita dabit æquationem pro curva quæſita. Habebitur igitur iſta æquatio $(\alpha + \gamma) dA + \beta dB = 0$, ſeu $(\alpha + \gamma) \delta dA + \beta \delta dB = 0$; in qua, etiamſi α & β ſint quantitates conſtantes determinatæ, tamen, ob γ & δ quantitates conſtantes arbitrarias, coëfficientes valorum dA & dB, qui ſunt $(\alpha + \gamma)\delta$ & $\beta\delta$ evadent conſtantes arbitrariæ magnitudinis. Harum igitur loco ſi ſcribantur litteræ ξ & η, habebitur pro curva quæſita iſta æquatio $\xi dA + \eta dB = 0$. Quocirca ad Problema ſolvendum, expreſſionum A & B, quarum altera proprietatem communem continet, utriuſque autem functio quæcunque maximum minimumve eſſe debet, ſingulatim valores differentiales dA & dB capi oportet, eoſque, per quantitates conſtantes arbitrarias, quaſque multiplicatos nihilo æquales poni, quo pacto reſultabit iſta æquatio $\xi dA + \eta dB = 0$, quæ naturam curvæ quæſitæ exprimet. Q. E. I.

COROLL. I.

68. Natura igitur curvæ ſatisfacientis tantum ab expreſſionibus A & B pendet; neque ratio functionis ipſarum A & B, quæ maximum minimumve eſſe debet, ullo modo in computo manet; ſed quæcunque ſit functio, eadem ſolutio prodibit.

COROLL. II.

69. Quæcunque itaque ipſarum A & B functio, inter omnes curvas eadem proprietate A gaudentes, debeat eſſe maximum vel minimum; ſolutio perinde ſe habebit, ac ſi, inter omnes curvas eadem communi proprietate A gaudentes, ea requiratur, in qua expreſſio altera B maximum minimumve obtineat valorem.

Co-

Digitized by Google

COROLL. III.

70. Quod ſi ergo expreſſiones A & B ejuſmodi fuerint formulæ, quarum valores differentiales dA & dB non pendeant a magnitudine abſciſſæ x cui reſpondent; quod evenit, ſi illæ formulæ pertineant ad Caſum vel primum vel quartum, ſecundum noſtram enumerationem Capite præcedente §. 7 factam, tum curva inventa pro quacunque abſciſſa æque ſatisfaciet.

COROLL. IV.

71. Eadem Solutio locum habebit ſi, inter omnes curvas quarum communis ſit proprietas functio quæcunque ipſarum A & B, ea requiratur in qua alia quæpiam earundem A & B functio ſit maximum vel minimum. Hoc enim quoque caſu pervenitur ad æquationem $\xi dA + \eta dB = 0$, in qua ξ & η ſint quantitates conſtantes ad arbitrium accipiendæ.

EXEMPLUM I.

72. *Inter omnes curvas* aMb *cum axe* AB *eandem aream* Fig. 10. $\int y\,dx$ *continentes, invenire eam in qua ſit* $\frac{\int yy\,dx}{\int y\,dx}$ *minimum.*

Quæſtio hæc initur, ſi inter omnes areas æquales quæ intra ordinatas extremas Aa & Bb atque baſi AB formari poſſunt, deſideretur ea, quæ habeat ſuum centrum gravitatis in loco infimo poſitum. Sumpta enim curva quacunque aMb, poſitiſque abſciſſa $AP = x$, applicata $PM = y$, erit portionis aAPM centrum gravitatis a baſi AP remotum intervallo $= \frac{\int yy\,dx}{2\int y\,dx}$; quod adeo fiet minimum, ſi reddatur hæc expreſſio $\frac{\int yy\,dx}{\int y\,dx}$ minima. Habemus ergo binas has formulas $\int y\,dx$ & $\int yy\,dx$, quarum valores differentiales ſunt $nv.\,dx.\,1$ & $nv.\,dx.\,2y$, ex

 qui-

Digitized by Google

quibus pro curva quæſita iſta colligitur æquatio $\xi + 2\eta y = 0$; ſeu $y = c$. Quæſtioni igitur ſatisfacit linea recta $\alpha\mathfrak{C}$ baſi AB parallela ſeu horizontalis, atque parallelogrammum rectangulum $A\alpha\mathfrak{C}B$, præ omnibus aliis figuris ut AabB ejuſdem areæ, hac gaudebit prærogativa, ut ejus centrum gravitatis ad baſin AB proxime accedat. Quod ſi ergo α AB$\mathfrak{C}$ concipiatur tanquam vas aqua repletum, ſi ſuprema aquæ ſuperficies $\alpha\mathfrak{C}$ ſeſe ad ſitum horizontalem compoſuerit, tum aqua habebit ſuum centrum gravitatis profundius ſitum, quam ſi ejus ſuprema ſuperficies alium quemcunque ſitum teneret.

EXEMPLUM II.

Fig. 14. 73. *Inter omnes curvas ejuſdem longitudinis* DAD, *invenire eam quæ habeat ſuum gravitatis centrum quam profundiſſime ſitum, ſeu in qua ſit* $\frac{\int x\,dx\,\surd(1+pp)}{\int dx\,\surd(1+pp)}$ *minimum.*

Jam intelligitur Solutio hujus Quæſtionis datura eſſe curvam Catenariam; namque ſecundum leges Staticas catena ex punctis D & D ſuſpenſa ejuſmodi induet figuram ut ejus centrum gravitatis maxime deſcendat. Quamobrem inter omnes figuras, quas catena inducere poteſt, quæ quidem omnes ejuſdem ſunt longitudinis, curva Catenaria orietur, ſi quæratur ea, in qua ſit $\frac{\int x\,dx\,\surd(1+pp)}{\int dx\,\surd(1+pp)}$ minimum; quippe quæ expreſſio dat diſtantiam centri gravitatis G ab abſciſſarum initio A. Cum igitur habeantur binæ iſtæ formulæ $\int dx\,\surd(1+pp)$, & $\int x\,dx\,\surd(1+pp)$; quærantur earum valores differentiales; qui erunt, primæ $= - n\,\nu.\,d.\,\frac{p}{\surd(1+pp)}$, & alterius $= - n\,\nu.\,d.\,\frac{xp}{\surd(1+pp)}$, ex quibus naſcitur pro curva quæſita iſta æquatio $c\,d.\,\frac{p}{\surd(1+pp)} = d.\,\frac{xp}{\surd(1+pp)}$, & integrando $\frac{xp}{\surd(1+pp)} = \frac{cp}{\surd(1+pp)}$ +

$+b$, ſeu $x-c=\frac{b\sqrt{(1+pp)}}{p}$, & $dx=\frac{-bdp}{pp\sqrt{(1+pp)}}$. Hinc ergo ſiet $y=\int p\,dx=-b\int\frac{dp}{p\sqrt{(1+pp)}}$; ex quibus æquationibus curva conſtruetur, eritque curvæ longitudo $\int dx\sqrt{(1+pp)}=s=\frac{b}{p}+Conſt.=\frac{b}{p}+f$. Hinc alia Conſtructio, definiendis x & y per s formari poterit: erit nempe $p=\frac{b}{s-f}$ &, ſi initium capiatur in A, ubi ſit $p=\infty$, ponendum eſt $f=0$, ita ut ſit $p=\frac{b}{s}$; unde ſit $\sqrt{(1+pp)}=\frac{\sqrt{(bb+ss)}}{s}$ & $dx\sqrt{(1+pp)}=ds=\frac{dx\sqrt{(bb+ss)}}{s}$; hincque $dx=\frac{s\,ds}{\sqrt{(bb+ss)}}$, & $x=\sqrt{(bb+ss)}-b$. Porro erit $dy=p\,dx=\frac{b\,ds}{\sqrt{(bb+ss)}}$, atque $y=bl\frac{s+\sqrt{(bb+ss)}}{b}$. Æquatio autem inter coordinatas orthogonales x & y deducetur ex æquatione $x-c=\frac{b\sqrt{(1+pp)}}{p}$; quæ ſi deſideretur ſuper axe AP, qui eſt diameter, & pro initio abſciſſarum in A ſumpto, ubi eſt $p=\infty$, poni oportet $c=-b$; eritque $(x+b)p=b\sqrt{(1+pp)}$, hincque $(x+b)^2pp=bb+bbpp$, & $p=\frac{b}{\sqrt{(xx+2bx)}}$; ideoque $dy=\frac{b\,dx}{\sqrt{(xx+2bx)}}$, quæ eſt æquatio pro Catenaria nota.

EXEMPLUM III.

74. *Inter omnes curvas ejuſdem longitudinis, determinare eam in qua ſit* $\frac{\int S\,x\,dx\sqrt{(1+pp)}}{\int S\,dx\sqrt{(1+pp)}}$ *minimum; denotante* S *functionem quamcunque arcus curvæ* $s=\int dx\sqrt{(1+pp)}$.

In hoc Exemplo continetur inventio curvæ Catenariæ, ſi catena non fuerit ubique uniformiter craſſa, ſed cujus craſſities ar-

cui s reſpondens eſt ut S functio ipſius s. Tum enim exprimet $\int S dx \sqrt{(1+pp)}$ hujus catenæ pondus, & $\frac{\int S x dx \sqrt{(1+pp)}}{\int S dx \sqrt{(1+pp)}}$ altitudinem centri gravitatis ſupra abſciſſarum initium; quæ eſſe debet minima. Principio quidem hic caſus in Problemate præcedente non contineri videtur, quia formula arcum exprimens ipſa $\int dx \sqrt{(1+pp)}$ non ineſt in maximi minimive expreſſione $\frac{\int S x dx \sqrt{(1+pp)}}{\int S dx \sqrt{(1+pp)}}$, quippe quæ eſt functio duarum aliarum formularum integralium. At cum ſit S functio arcus curvæ s, atque $ds = dx \sqrt{(1+pp)}$, erit $\int S dx \sqrt{(1+pp)} = \int S ds$, ideoque functio ipſius s: ex quo expreſſio $\frac{\int S x dx \sqrt{(1+pp)}}{\int S dx \sqrt{(1+pp)}}$, erit functio formularum $\int dx \sqrt{(1+pp)}$ & $\int S x dx \sqrt{(1+pp)}$, quarum illa proprietatem communem continet. Idem igitur eſt ac ſi quærere deberemus inter omnes curvas æque longas eam in qua ſit $\int S x dx \sqrt{(1+pp)}$ minimum. Cum jam S ſit functio ipſius $s = \int dx \sqrt{(1+pp)}$, pertinebit hæc Quæſtio ad Propoſitionem præcedentem, eumque caſum qui §. 60 eſt pertractatus. Scilicet erit $Z = Sx \sqrt{(1+pp)}$; unde, ſi ponamus $dS = T ds$, fiet $dZ = xT ds \sqrt{(1+pp)} + S dx \sqrt{(1+pp)} + \frac{S x p dp}{\sqrt{(1+pp)}}$, ita ut ſit $L = xT \sqrt{(1+pp)}$; $M = S \sqrt{(1+pp)}$, $N = 0$ & $P = \frac{Sxp}{\sqrt{(1+pp)}}$. Jam ob $N = 0$, obtinemus ex eodem loco citato ſtatim hanc æquationem . . . .
$A + \frac{(C - \int x T dx \sqrt{(1+pp)})p}{\sqrt{(1+pp)}} = \frac{-Sxp}{\sqrt{(1+pp)}}$, ſeu $\frac{A\sqrt{(1+pp)}}{p} + C - \int x T dx \sqrt{(1+pp)} + Sx = 0$. At eſt $T dx \sqrt{(1+pp)} = T ds = dS$; unde habetur $\frac{A\sqrt{(1+pp)}}{p} + C + Sx - \int x dS = 0$, ubi A & C ſunt quantitates arbitrariæ. Differentietur hæc æquatio, fietque $\frac{-A dp}{pp\sqrt{(1+pp)}} + S dx = 0$, ſeu $S dx \sqrt{(1+pp)} = -\frac{c dp}{pp} = S ds$. Quare cum ſit S functio ipſius

ipſius s, integretur $S\,ds$, eritque integrale, quod ſit $= R$, pondus catenæ longitudini s reſpondens. Fiet ergo integrando $\frac{c}{p} = R + C$; &, ſi initium curvæ capere placeat in loco A, ubi curvæ tangens eſt horizontalis, erit $C = 0$, atque $p = \frac{c}{R}$. Hinc ergo porro erit $\sqrt{(1+pp)} = \frac{\sqrt{(cc+RR)}}{R} = \frac{ds}{dx}$; ideoque $dx = \frac{R\,ds}{\sqrt{(cc+RR)}}$, atque $dy = \frac{c\,ds}{\sqrt{(cc+RR)}}$, ex quibus æquationibus curva ita poterit conſtrui, ut ſtatim ad quamvis catenæ longitudinem tam abſciſſa quam applicata reſpondens definiatur. Manifeſtum autem eſt caſu quo $R = s$, hoc eſt quo catena ponitur uniformis craſſitiei, tum prodire Catenariam curvam ordinariam.

SCHOLION.

75. Niſi hujus Exempli convenientia, tam cum iſta Propoſitione quam cum præcedente, eſſet obſervata, tum Solutio quidem per regulam generalem abſolvi potuiſſet: verum tamen multo prolixior evaſiſſet. Quo autem nihilominus Methodi generalis uſus clarius ob oculos ponatur, idem hoc Exemplum ſecundum generalia præcepta reſolvere viſum eſt. Quæratur igitur inter omnes curvas ejuſdem longitudinis $s = \int dx \sqrt{(1+pp)}$, ea quæ habeat valorem expreſſionis hujus $\frac{\int Sx\,dx\sqrt{(1+pp)}}{\int S\,dx\sqrt{(1+pp)}}$ maximum vel minimum; exiſtente S functione quacunque arcus curvæ s. Et quoniam nondum ſuſpicari licet conſiderationem datæ abſciſſæ, a qua valor differentialis expreſſionis $\frac{\int Sx\,dx\sqrt{(1+pp)}}{\int S\,dx\sqrt{(1+pp)}}$ pendet, ex calculo eſſe egreſſuram; ponamus huic Quæſtioni tantum pro data abſciſſæ longitudine $x = a$ ſatisfieri oportere. Ab hac longitudine quidem formulæ communem proprietatem continentis $\int dx \sqrt{(1+pp)}$ valor differentialis non pendet, quippe

Digitized by Google

quippe qui conſtanter eſt $= - nv.\ d.\ \frac{p}{\sqrt{(1+pp)}}$; at in maximi minimive expreſſione $\frac{\int Sxdx\sqrt{(1+pp)}}{\int Sdx\sqrt{(1+pp)}}$ ponamus, caſu quo $x=a$, fore $\int Sxdx\sqrt{(1+pp)}=A$ & $\int Sdx\sqrt{(1+pp)} = B$: illius vero numeratoris $\int Sx\,dx\sqrt{(1+pp)}$ valorem differentialem eſſe $=dA$, denominatoris vero $\int S\,dx\sqrt{(1+pp)}$ valorem differentialem eſſe $=dB$. Hinc igitur maximi minimive expreſſionis, quæ, caſu $x=a$, fit $=\frac{A}{B}$, valor differentialis erit $\frac{BdA - AdB}{BB}$, qui multiplo cuicunque formulæ communis valoris differentialis $- nv.\ d.\ \frac{p}{\sqrt{(1+pp)}}$ æqualis poſitus dabit æquationem pro curva quæſita. Jam ad valores differentiales dA & dB inveniendos, conſideremus primum formulam $\int Sdx\sqrt{(1+pp)}$, quæ ſecundum enumerationem §. 7 Cap. præced. factam, pertinet ad Caſum ſecundum: quo erit $Z=S\sqrt{(1+pp)}$, & poſito $dS=Tds$, erit $dZ = Tds\sqrt{(1+pp)}+\frac{Spdp}{\sqrt{(1+pp)}}$. Comparatione ergo facta, erit $\Pi=s$; $L=T\sqrt{(1+pp)}$, $M=0$, $N=0$, & $P=\frac{Sp}{\sqrt{(1+pp)}}$: tum vero ob $\Pi=s=\int dx\sqrt{(1+pp)}$, erit $[Z]=\sqrt{(1+pp)}$ & $[M]=0$, $[N]=0$, & $[P]=\frac{p}{\sqrt{(1+pp)}}$. Jam ſumatur integrale $\int Ldx=\int Tdx\sqrt{(1+pp)}=\int Tds=S$, cujus valor, caſu $x=a$, fiat $=G$, eritque $V=G-S$. Quamobrem habebitur formulæ $\int Sdx\sqrt{(1+pp)}$ valor differentialis $dB = - nv.\ d(\frac{Sp}{\sqrt{(1+pp)}}+\frac{p(G-S)}{\sqrt{(1+pp)}}) = - nv.\ d.\ \frac{Gp}{\sqrt{(1+pp)}}$. Altera porro formula $\int Sxdx\sqrt{(1+pp)}$ pariter in eodem Caſu ſecundo comprehenditur, eritque $Z=Sx\sqrt{(1+pp)}$ & $dZ=Txds\sqrt{(1+pp)}+Sdx\sqrt{(1+pp)}+\frac{Sxpdp}{\sqrt{(1+pp)}}$ unde fit $\Pi=s$; $L=Tx\sqrt{(1+pp)}$; $M=S\sqrt{(1+pp)}$;

$N=0$

$N = 0$ & $P = \frac{Sxp}{\surd(1+pp)}$. Deinde, ob $\Pi = s = \int dx\surd(1+pp)$, erit, ut ante, $[Z] = \surd(1+pp)$, $[M] = 0$, $[N] = 0$, & $[P] = \frac{p}{\surd(1+pp)}$. Nunc ſumatur integrale $\int L\,dx = \int Tx\,dx\surd(1+pp) = \int Tx\,ds = \int x\,dS$, cujus valor, poſito $x = a$, ſit $= H$; erit $V = H - \int x\,dS$, hincque prodibit iſtius formulæ valor differentialis $dA = \ldots\ldots$ $-nv.d(\frac{Sxp+p(H-\int xdS)}{\surd(1+pp)}) = -nv.d.\frac{p(H+\int Sdx)}{\surd(1+pp)}$. Inventis ergo valoribus dA & dB, æquatio pro curva quæſita erit $\alpha B^2 d.\frac{p}{\surd(1+pp)} - \beta A d.\frac{Gp}{\surd(1+pp)} + \beta B d.\frac{p(H+\int Sdx)}{\surd(1+pp)} = 0$, & integrando $\frac{\alpha B^2 p - \beta AGp + \beta BHp + \beta Bp\int Sdx}{\surd(1+pp)} = C$; in qua α, β, & C ſunt conſtantes arbitrariæ, & G & H conſtantes determinatæ. Quod ſi ergo ponatur $\frac{\alpha B}{\beta} + \frac{AG}{B} + H = b$ & $\frac{C}{\beta B} = c$, erunt b & c conſtantes arbitrariæ, atque conſtantes determinatæ G & H a definito abſciſſæ valore $x = a$ pendentes omnino ex æquatione evaneſcent; ita ut Curva inventa pro quavis abſciſſa gaviſura ſit deſiderata proprietate: ejuſque æquatio erit hæc $c = \frac{bp + p\int Sdx}{\surd(1+pp)}$, ſeu $\frac{c\surd(1+pp)}{p} = b + \int Sdx$; quæ differentiata dabit $Sdx = -\frac{cdp}{pp\surd(1+pp)}$, ſeu $Sdx\surd(1+pp) = Sds = -\frac{cdp}{pp}$. Ponatur, ut ſupra, $\int Sds = R$, ita ut R pondus longitudinis catenæ s repræſentet, erit $R = \frac{c}{p} + Conſt.$ quæ eſt ipſa æquatio, quam præcedenti Methodo elicuimus. Ex hac itaque ſolutione intelligitur, quemadmodum per Methodum generalem hujuſmodi Quæſtiones reſolvi poſſint, ſi proprietas communis non ingrediatur in maximi minimive expreſſionem; quod ut clarius intelligatur unum adhuc hujuſmodi Exemplum appoſuiſſe ſufficiet.

Digitized by Google

EXEMPLUM IV.

Fig. 14. 76. *Inter omnes curvas ejuſdem longitudinis* DAD *data abſciſſa* AC $=$ a *reſpondentes, eam definire quæ comprehendat aream* DAD, *cujus centrum gravitatis* G *ſit vel altiſſime vel profundiſſime poſitum, ſeu in qua ſit* $\frac{\int y\,x\,dx}{\int y\,dx}$ *maximum vel minimum.*

Proprietas igitur communis eſt $\int dx \sqrt{(1+pp)}$, cujus valor differentialis cuicunque abſciſſæ x reſpondens eſt $= - nv.\, d.\, \frac{p}{\sqrt{(1+pp)}}$. Maximi autem minimive expreſſionis $\frac{\int yx\,dx}{\int y\,dx}$ valor differentialis pendebit a præſcripta abſciſſæ longitudine $x = a$; qui ut inveniatur; caſu quo $x = a$, fiat $\int yx\,dx = A$, hujuſque formulæ valor differentialis ſit $= dA$, qui per Regulas ſupra datas invenitur $= nv.\,dx.\,x = nv.\,x\,dx$. Porro, eodem caſu $x = a$, abeat altera formula $\int y\,dx$ in B, ſitque ejus valor differentialis $= dB$, qui per Regulas datas reperitur $= nv.\,dx$; ita ut ſit $dA = nv.\,x\,dx$ & $dB = nv.\,dx$. Ex his, maximi minimive expreſſionis $\frac{\int yx\,dx}{\int y\,dx}$, quæ, caſu $x = a$, abit in $\frac{A}{B}$, valor differentialis erit $= \frac{B\,dA - A\,dB}{BB} = \frac{nv\,(B\,x\,dx - A\,dx)}{BB}$, qui multiplo valoris differentialis $- nv.\,d.\,\frac{p}{\sqrt{(1+pp)}}$, qui ex proprietate communi prodiit, æqualis poſitus dabit pro curva quæſita iſtam æquationem $\alpha\,d.\,\frac{p}{\sqrt{(1+pp)}} = \frac{B\,x\,dx - A\,dx}{BB}$. Sit $\frac{A}{B} = h$, erit h quantitas conſtans determinata, quam præbet formula $\frac{\int yx\,dx}{\int y\,dx}$, ſi ponatur $x = a$, & αB ponatur $= cc$, erit cc quantitas arbitraria. Hinc habebitur iſta pro curva æquatio $cc\,d.\,\frac{p}{\sqrt{(1+pp)}} = x\,dx - h\,dx$, quæ

Digitized by Google

quæ integrata dat $\frac{2ccp}{\sqrt{(1+pp)}} = xx - 2bx + bb$; ergo $4c^4pp = (xx - 2bx + bb)^2(1+pp)$, atque $p = \frac{xx - 2bx + bb}{\sqrt{(4c^4 - (xx - 2bx + bb)^2)}} = \frac{dy}{dx}$. Quocirca erit $y = \int \frac{(xx - 2bx + bb)dx}{\sqrt{(4c^4 - (xx - 2bx + bb)^2)}}$, ubi conſtantem bb, pro arbitrio, ſive affirmativam, ſive negativam accipere licet. Hæc autem curva Quæſtioni ſatisfacit tantum caſu, quo $x = a$; atque ut ſatisfaciat litteræ b is tribui debet valor quem, caſu $x = a$, recipiet expreſſio $\frac{\int yx\,dx}{\int y\,dx}$, ex quo valor b determinabitur. Cæterum notari convenit hanc curvam eſſe eam quæ vulgo ſub nomine Elaſticæ eſt cognita.

CAPUT VI.

Methodus, inter omnes curvas pluribus proprietatibus communibus gaudentes, eam determinandi quæ maximi minimive proprietate ſit prædita.

Propositio I. Theorema.

1. *Curva, quæ inter omnes omnino curvas habet expreſſionem* $\alpha A + \beta B$ *maximum vel minimum, eadem ſimul ita erit comparata, ut inter omnes eadem proprietate* A *præditas contineat valorem formulæ* B *maximum vel minimum.*

DEMONSTRATIO.

Ponamus inventam eſſe curvam, in qua inter omnes alias eidem abſciſſæ reſpondentes valor expreſſionis $\alpha A + \beta B$ ſit maximus; quod enim de maximo demonſtrabitur, idem mutatis mutandis de minimo valebit. Denotant autem litteræ A & B hic nobis ejuſmodi formulæ vel expreſſiones indeterminatæ, in quas Quæſtio de maximis & minimis cadere queat; tum vero

α & β ſunt quantitates conſtantes quæcunque. Deſignemus jam iſtam curvam, in qua ſit $\alpha A + \beta B$ maximum, littera Q, quo eam facilius ſine moleſta verborum deſcriptione indicare queamus; Nunc concipiatur alia quæcunque curva R eidem abſciſſæ reſpondens, quæ recipiat formulæ A eundem valorem, quem tenet curva Q: in hac igitur curva R expreſſio $\alpha A + \beta B$ minorem occupabit valorem, quam in curva Q; eo quod in curva Q expreſſio $\alpha A + \beta B$ omnium maximum valorem ſortitur. Quare cum in curvis Q & R expreſſio A eundem obtineat valorem, atque in Q expreſſio $\alpha A + \beta B$ major ſit quam in curva R; ſequitur in curva Q valorem expreſſionis B majorem eſſe debere quam in curva R. Cum igitur R curvam quamcunque denotet, quæ cum Q communem valorem formulæ A recipiat; manifeſtum eſt inter omnes has curvas R curvam Q eſſe illam, in qua formula B maximum habeat valorem. Ex quibus conficitur, eam curvam, quæ inter omnes omnino curvas habeat expreſſionis $\alpha A + \beta B$ valorem maximum vel minimum, eandem curvam ſimul ita eſſe comparatam, ut inter omnes alias curvas ſecum eadem communi proprietate A gaudentes poſſideat maximum minimumve valorem expreſſionis B. Quanquam enim Demonſtratio tantum ad maximum eſt adornata, tamen eadem, translatis verbis, ad minimum accommodabitur. Q. E. D.

Coroll. I.

2. Viciſſim itaque intelligitur, ſi curva debeat inveſtigari, quæ inter omnes alias eadem communi proprietate A præditas expreſſionem B ſit habitura maximum vel minimum; tum quæſito ſatisfieri, ſi abſolute inter omnes curvas ea definiatur, in qua ſit $\alpha A + \beta B$ maximum vel minimum.

Coroll. II.

3. In ſolutionem igitur hujuſmodi Problematum binæ novæ ingrediuntur conſtantes arbitrariæ α & β, quæ in ipſis expreſſionibus

Digitized by Google

ſtibus A & B non inerant: hæ autem unius dumtaxat conſtantis vicem ſuſtinebunt; quia earum ratio tantum in computum venit.

COROLL. III.

4. Quod ſi ergo, inter omnes curvas eadem communi proprietate A gaudentes, eam definiri oporteat in qua ſit B maximum minimumve; tum utriuſque expreſſionis A & B capiantur valores differentiales, qui, per conſtantes arbitrarias ſeorſim multiplicati & conjunctim nihilo æquales poſiti, dabunt æquationem pro curva quæſita.

COROLL. IV.

5. Simul etiam perſpicuum eſt perinde eſſe, ſive inter omnes curvas eadem communi proprietate A gaudentes, ea quæratur in qua ſit B maximum vel minimum; ſive viciſſim inter omnes curvas eadem communi proprietate B gaudentes, ea quæratur in qua ſit A maximum vel minimum.

SCHOLION.

6. Quæ, cum in hac Propoſitione, tum in annexis Corollariis, tradidimus, ex Capite præcedente jam ſunt planiſſima: quippe quibus continetur inverſa Methodus reſolvendi Problemata, in quibus, inter omnes curvas eadem communi proprietate gaudentes, ea quæritur quæ prædita ſit maximi minimive alicujus indole. Neque vero idcirco idem argumentum nos tantum repetiviſſe cenſendum eſt; nam eandem veritatem, quam ante modo ſatis prolixo elicueramus, hîc admodum ſuccincte & breviter dedimus demonſtratam. Quocirca eo fortius altera demonſtrandi Methodus per alteram confirmabitur, ob ſummum utriuſque conſenſum: atque ſi cui prior Methodus non ſatis perſpecta, propter tantam infinite parvorum compagem, nimis lubrica & incerta videatur, ei Demonſtratio hîc data omnem

Digitized by Google

nem ſcrupulum adimet. Deinde, ſi quis de præſentis Propoſitionis converſione in Coroll. 1 facta etiamnum dubitet, ei prior Methodus pleniſſime ſatisfaciet. Interim ratio converſionis ex ſe ſatis tuto inferri poteſt. Cum enim curva *Q*, quæ inter omnes omnino curvas habeat $\alpha A + \beta B$ maximum vel minimum, ita ſit comparata, ut inter omnes curvas eadem communi proprietate *A* gaudentes, habeat *B* maximum vel minimum, quicquid loco α & β accipiatur; neceſſe eſt ut converſio æque pateat, ſiquidem coëfficientibus α & β ſumma extenſio tribuatur. Hocque adeo commemorare, hujuſque ratiocinii validitatem declarare viſum eſt, ut in ſequentibus, ubi eodem utemur, nullum dubium relinquatur. Hanc enim Propoſitionem, etſi proprie ad Caput præcedens pertinet, huc tranſtulimus, quo eâdem Methodo proprium hujus Capitis argumentum facilius pertractare poſſimus; quippe quod, ſi altera Methodo expediri deberet, prolixiſſimos requireret calculos, maximaſque differentialium omnium ordinum tricas. Interim tamen, quantum fieri poteſt, dilucide oſtendemus omnia, quæ hîc trademus, per Methodum ſuperiorem confirmari atque etiam elici poſſe.

PROPOSITIO II. THEOREMA.

7. *Quæ curva, inter omnes omnino curvas eidem abſciſſæ reſpondentes, habet valorem expreſſionis* $\alpha \mathrm{A} + \beta \mathrm{B} + \gamma \mathrm{C}$ *maximum vel minimum, eadem curva ſimul ita erit comparata, ut inter omnes curvas, quæ tam expreſſionem* A *quam expreſſionem* B *communem habeant, poſſideat valorem expreſſionis* C *maximum vel minimum.*

DEMONSTRATIO.

Denotant hîc nobis litteræ *A*, *B* & *C* formulas integrales vel expreſſiones indefinitas ejuſmodi, quæ maximi minimive ſint capaces; at litteræ α, β, γ deſignant quantitates conſtantes arbitrarias. Sit nunc *Q* curva, quæ inter omnes omnino curvas habeat

Digitized by Google

habeat valorem $\alpha A + \beta B + \gamma C$ maximum vel minimum; atque concipiatur alia quæcunque curva R, in qua, cum expreſſio A, tum B, eundem obtineat valorem quem obtinet in curva Q; quo poſito expreſſio compoſita $\alpha A + \beta B$ eundem habebit valorem in utraque curva Q & R. Hanc ob rem, expreſſio tota $\alpha A + \beta B + \gamma C$ in curva R minorem ſortietur valorem quam in curva Q, ſiquidem $\alpha A + \beta B + \gamma C$ in curva Q eſt maximum; contra expreſſionis $\alpha A + \beta B + \gamma C$ valor in curva R major erit quam in curva Q, ſi $\alpha A + \beta B + \gamma C$ in curva Q fuerit minimum. Cum igitur expreſſionis portio $\alpha A + \beta B$ utrique curvæ Q & R ſit communis, reliqua portio γC, atque adeo expreſſio C, in caſu maximi major erit in Q quam in R, in caſu minimi autem expreſſio C in curva Q minor erit quam in curva R. Ex quibus ſequitur, ſi curva Q inter omnes omnino curvas, habuerit valorem expreſſionis $\alpha A + \beta B + \gamma C$ maximum vel minimum, tum ſimul hanc curvam Q ea indole eſſe præditam, ut inter omnes curvas R quæ eodem valore cum expreſſionis A tum expreſſionis B gaudeant, contineat valorem expreſſionis C maximum vel minimum. Q. E. D.

COROLL. I.

8. Quoniam expreſſiones A, B & C pro lubitu inter ſe commutari poſſunt; curva in qua eſt $\alpha A + \beta B + \gamma C$ maximum vel minimum, ea ſimul vel, inter omnes curvas iiſdem proprietatibus A & B communibus gaudentes, habebit C maximum vel minimum; vel habebit B maximum minimumve, inter omnes curvas quæ proprietatibus A & C communibus gaudebunt; vel denique habebit A maximum minimumve, inter omnes curvas in quas ambæ proprietates B & C æque competunt.

COROLL. II.

9. Quæ igitur curva, inter omnes iiſdem binis proprietatibus A & B communibus gaudentes, habet C maximum minimum-

mumve; eadem habebit inter omnes curvas binis proprietatibus vel A & C, vel B & C, æque præditas, vel B, vel A maximum minimumve.

Coroll. III.

10. Si igitur curva quæri debeat quæ, inter omnes alias binis proprietatis A & B æqualiter præditas, habeat expreſſionem C maximam vel minimam; tum quæſito ſatisfiet, ſi curva quæratur, quæ, abſolute inter omnes curvas, habeat expreſſionem $\alpha A + \beta B + \gamma C$ maximum vel minimum.

Coroll. IV.

11. Quoniam α, β, γ ſunt quantitates conſtantes arbitrariæ; in ſolutionem hujuſmodi Problematum tres novæ quantitates arbitrariæ ingrediuntur, quæ in formulis propoſitis A, B, & C non inerant: æquivalebunt autem hæ tres conſtantes α, β, & γ tantum duabus.

Coroll. V.

12. Hæ vero conſtantes adeo jam in æquatione pro curva primum inventa inerant; præter eas vero, per integrationes novæ ingredientur conſtantes tot, quot integrationibus opus eſt, antequam ad æquationem finitam perveniatur.

Coroll. VI.

13. Simili modo, quo hanc Propoſitionem & præcedentem demonſtravimus, oſtendetur, curvam, quæ abſolute, inter omnes curvas, habeat expreſſionem $\alpha A + \beta B + \gamma C + \delta D$ maximam vel minimam, eandem, inter omnes curvas tres expreſſiones A, B & C communes habentes, habituram eſſe quartam D maximam vel minimam.

Digitized by Google

SCHOLION.

14. Ex hac Propoſitione jam ſatis percipitur Methodus reſolvendi ejuſmodi Problemata ad Methodum relativam pertinentia, in quibus quæritur curva quæ, inter omnes eidem abſciſſæ reſpondentes & duabus pluribuſve proprietatibus communibus æque gaudentes, habeat valorem cujuſpiam expreſſionis maximum minimumve. Quæſtio ſcilicet perpetuo revocabitur ad Methodum abſolutam; ita ut, inter omnes omnino curvas, quærenda ſit curva quæ expreſſionem quampiam habeat maximam vel minimam. Hacque reductione id commodi nanciſcimur, ut omnia hujuſmodi Problemata, ope valorum differentialium quos jam ſupra inveſtigare docuimus, reſolvere queamus. Ipſe autem reſolvendi modus eo redibit, ut omnes proprietates communes, una cum maximi minimive expreſſione, ſeorſim explicentur; ſingulæ per conſtantes arbitrarias multiplicentur; & producta in unam ſummam colligantur: quo facto, abſolute inter omnes curvas, eam quæri oportebit, in qua iſta ſumma ſit maxima vel minima. Hoc vero ipſum perficietur, dum ſummæ illius valor differentialis inveſtigabitur, nihiloque æqualis ponetur. Quocirca univerſa operatio abſolvetur, ſi, cum ſingularum expreſſionum proprietates communes continentium, tum maximi minimive expreſſionis valores differentiales, ſecundum regulas ſupra datas, capiantur; ſinguli ſeorſim in conſtantes arbitrarias ducantur; omniumque horum productorum aggregatum nihilo æquale ponatur: ex quo orietur æquatio pro curva quæſita. Sufficere itaque poſſet hoc unicum præceptum ad Quæſtiones hujus generis ſolvendas. Verum, antequam hujus uſum exponamus, hanc ipſam Methodum via ante adhibita confirmari conveniet.

PROPOSITIO III. PROBLEMA.

15. *Inter omnes curvas ad eandem abſciſſam relatas, quæ binis proprietatibus communibus* A *&* B *æqualiter ſint prædita; definire eam, in qua ſit valor expreſſionis* C *maximus vel minimus.*

Digitized by Google

SOLUTIO.

Ex præcedentibus jam intelligitur hoc Problema ſolvi, ſi, inter omnes curvas, abſolute quæratur ea in qua ſit $\alpha A + \beta B + \gamma C$ maximum vel minimum. Ad hoc autem noſſe oportet valores differentiales expreſſionum A, B, & C. Sit igitur valor differentialis expreſſionis $A = n\nu . dx . P$; expreſſionis $B = n\nu . dx . Q$; expreſſionis $C = n\nu . dx . R$; ex quibus æquatio pro curva deſiderata erit $\alpha P + \beta Q + \gamma R = 0$.

Verum, quo hujus Solutionis veritas magis eluceat, idem hoc Problema eadem Methodo, qua ſupra in Capite præced. uſi Fig. 15. ſumus, aggrediamur. Primum autem intelligitur, ad hoc Problema reſolvendum, ternas applicatas particulis infinite parvis augeri debere, ut tribus conditionibus præſcriptis ſatisfieri poſſit. Primo enim tres has particulas adjunctas, quibus ipſa curva ſatisfaciens az in novam a ſe quam-minime diſcrepantem transmutatur, ita comparatas eſſe oportet, ut expreſſio A, quæ unam proprietatem communem continet, in utramque curvam æqualiter competat: Deinde etiam altera proprietas communis B in utraque curva eundem valorem obtinere debebit. Tertio ex maximi minimive natura expreſſio quoque C eundem valorem in ipſa curva & eadem mutata nanciſci debet; quibus tribus conditionibus, per pauciores quam tres particulas tribus applicatis adjunctas, ſatisfieri non poteſt. Quare præter binas applicatas Nn & Oo, quæ in figura particulis $n\nu$ & $o\omega$ ſunt auctæ, concipiatur ſequenti applicatæ Pp particula $p\pi$ adjici. Ac quæratur primum incrementum, quod expreſſio A ex his tribus particulis aſſequitur quod erit $= n\nu . P dx + o\omega . P' dx + p\pi . P'' dx$. Namque ex particula $n\nu$ naſcitur incrementum $n\nu . P dx$, congruens cum ipſo valore differentiali, quem expreſſio A ex ſola particula $n\nu$ adipiſcitur. Ex ſequenti vero particula $o\omega$ oritur incrementum $o\omega . P' dx$, ſcilicet, idem quod ante, ſuo differentiali auctum: quia enim $o\omega$ ſequenti applicatæ adjungitur, omnes quantitates $o\omega$ afficientes erunt ſequentes earum, quibus particula $n\nu$ afficitur: atque ſimili ratione ex particula $p\pi$ naſcetur incre-

incrementum $p\pi.\ P''dx$; quæ omnia, ſi cui libuerit calculum eo modo quo in Cap. præc. Propoſ. 3 §. 22 uſi ſumus perſequi, ſatiſfient manifeſta ac perſpicua. Eodem igitur porro modo expreſſio B, cujus valorem differentialem ex unica particula $n\nu$ oriundum poſuimus $= n\nu.\ Qdx$, ex tribus particulis $n\nu$, $o\omega$ & $p\pi$ incrementum accipiet $= n\nu.\ Qdx + o\omega.\ Q'dx + p\pi.\ Q''dx$. Tertio expreſſio C ex his tribus particulis augmentum capiet hoc $n\nu.\ Rdx + o\omega.\ R'dx + p\pi.\ R''dx$. Singula jam hæc tria incrementa ſeorſim nihilo æqualia poni oportet, ut omnibus conditionibus præſcriptis ſatisfiat; unde tres ſequentes æquationes orientur, facta diviſione per dx,

$$o = n\nu.\ P + o\omega.\ P' + p\pi.\ P''$$
$$o = n\nu.\ Q + o\omega.\ Q' + p\pi.\ Q''$$
$$o = n\nu.\ R + o\omega.\ R' + p\pi.\ R''$$

Quod ſi nunc particulæ $n\nu$, $o\omega$, $p\pi$, ad ſolutionem peragendam tantum in ſubſidium vocatæ eliminentur; orietur æquatio inter quantitates curvæ proprias, quibus proin natura curvæ exprimetur. Ad has autem particulas eliminandas, ſingulas æquationes per novas incognitas α, β, γ, ſeorſim multiplicemus, ut habeatur

$$o = n\nu.\ \alpha P + o\omega.\ \alpha P' + p\pi.\ \alpha P''$$
$$o = n\nu.\ \beta Q + o\omega.\ \beta Q' + p\pi.\ \beta Q''$$
$$o = n\nu.\ \gamma R + o\omega.\ \gamma R' + p\pi.\ \gamma R'',$$

atque formentur hinc iſtæ æquationes,

$$o = \alpha P + \beta Q + \gamma R$$
$$o = \alpha P' + \beta Q' + \gamma R'$$
$$o = \alpha P'' + \beta Q'' + \gamma R''.$$

Hic ſtatim patet, ſi pro α, β, γ accipiantur quantitates conſtantes, tum primam æquationem reliquas binas ultro in ſe complecti; ſi enim fuerit $o = \alpha P + \beta Q + \gamma R$, tum ſimul erit $o = \alpha dP + \beta dQ + \gamma dR$, & $o = \alpha ddP + \beta ddQ + \gamma ddR$;

& quia eſt $P' = P + dP$; $Q' = Q + dQ$, $R' = R + dR$, atque $P'' = P + 2dP + ddP$; $Q'' = Q + 2dQ + ddQ$ & $R'' = R + 2dR + ddR$; fiet quoque

$$o = \alpha P' + \beta Q' + \gamma R'$$
&
$$o = \alpha P'' + \beta Q'' + \gamma R''.$$

Quocirca ad Problema ſolvendum formanda eſt hæc æquatio

$$o = \alpha P + \beta Q + \gamma R;$$

quæ, ſi loco α, β, & γ quantitates quæcunque conſtantes arbitrariæ ſcribantur, exprimet naturam curvæ quæſitæ. Congruit autem omnino hæc æquatio cum ea quam altera Methodo elicuimus, alteraque Methodus per alteram confirmatur. Q. E. I.

Coroll. I.

16. Omnia ergo hujus quoque generis Problemata reſolvi poſſunt, ope valorum differentialium ex unius applicatæ mutatione oriundorum, quos ſupra ſatis ampliter invenire docuimus.

Coroll. II.

17. Manifeſtum igitur eſt, ſi curva debeat inveniri quæ, inter omnes alias ad eandem abſciſſam relatas atque in quas binæ expreſſiones A & B æqualiter competant, habeat valorem expreſſionis C maximum minimumve; tum quæſtionem redire ad hanc, quæ ad Methodum abſolutam pertineat, ut, inter omnes omnino curvas ad eandem abſciſſam relatas, determinetur ea in qua ſit expreſſio $\alpha A + \beta B + \gamma C$ maximum vel minimum.

Coroll. III.

18. Simul vero etiam hinc Methodus patet reſolvendi Problema-

Digitized by Google

blemata, in quibus, inter omnes curvas in quas plures duabus atque adeo quotcunque proprietates æqualiter conveniant, ea requiritur quæ maximi minimive cujuſdam proprietate gaudeat.

Coroll. IV.

19. Quod ſi enim, inter omnes curvas in quibus expreſſiones A, B, C, D æquales obtineant valores, ea debeat inveſtigari in qua ſit expreſſio E maximum vel minimum; tum quæſito ſatisfiet, ſi inter omnes omnino curvas ea quæratur in qua ſit $\alpha A + \beta B + \gamma C + \delta D + \varepsilon E$ maximum vel minimum; denotantibus litteris α, β, γ, δ, ε quantitates quaſcunque conſtantes & arbitrarias.

Coroll. V.

20. Quo plures igitur proponantur proprietates, quæ iis curvis, ex quibus quæſitam maximi minimive indole præditam indagare oportet, communes eſſe debeant; eo plures in æquationem pro curva ingredientur quantitates conſtantes arbitrariæ; atque adeo eo plures curvæ ſatisfacientes in ea comprehendentur.

SCHOLION I.

21. Cur eo plures conſtantes in Solutionem ingrediantur, quo plures proponantur proprietates communes, ex præcedentibus facile colligi poteſt. Ponamus enim, inter omnes curvas eadem proprietate A gaudentes, eam inveſtigari oportere, in qua ſit B maximum vel minimum: ac primo quidem conſtabit huic Quæſtioni eam curvam eſſe ſatisfacturam quæ, inter omnes omnino curvas, habeat B maximum vel minimum; hæc enim, inter omnes quoque illas quæ ſecum eadem communi proprietate A gaudebunt, habebit B maximum vel minimum. Deinde autem licet innumerabilia iſtiuſmodi curvarum genera concipi, quæ ſingula eundem valorem expreſſionis A recipiant; in uno

Digitized by Google

quoque vero genere una erit curva, quæ præ reliquis valorem expreſſionis *B* contineat maximum vel minimum. Neceſſe autem eſt has curvas ſatisfacientes omnes in Solutione generali contineri debere. Cum igitur, ob unam proprietatem communem præſcriptam, numerus curvarum ſatisfacientium fiat infinitus, multo magis is augebitur, propter eandem rationem, ſi plures proprietates communes proponantur. Interim tamen ſi valores, quos habent ſingulæ proprietates communes in curvis ex quibus quæſitam erui oportet actu definiantur, tum utique ſolutio unicam Curvam ſatisfacientem præbebit. Conſtantes ſcilicet illæ eo inſervient ut valores, quos proprietates communes in curva inventa obtinebunt, pro arbitrio determinentur; ſic, per has conſtantes, in caſu duarum proprietatum communium *A* & *B*, curva poterit aſſignari quæ datos expreſſionum *A* & *B* recipiat valores, atque inſuper ita ſit comparata ut, inter omnes infinitas alias eoſdem illarum expreſſionum *A* & *B* valores recipientes, habeat valorem alius cujuſcunque expreſſionis *C* maximum vel minimum. Atque hæc eadem admonitio locum habet, ſi plures proprietates communes fuerint præſcriptæ; ex quo ſatis perſpicuum eſt, quid hiſce conſtantibus in Solutionem ingredientibus ſit faciendum, & quomodo eas ad uſum traduci oporteat: id quod in ſequentibus Exemplis clarius declarari poterit.

EXEMPLUM I.

Fig. 14. 22. *Inter omnes curvas ad eandem abſciſſam* AC $=a$ *relatas, quæ cum inter ſe ejuſdem ſint longitudinis, tum etiam æquales areas* DAD *comprehendant; determinare eam, quæ circa axem* AC *rotata generet ſolidum maximæ vel minimæ capacitatis.*

Poſitis abſciſſa AP $=x$, applicata PM $=y$ & $dy=pdx$; binæ proprietates communes propoſitæ ſunt $\int ydx$ & $\int dx\surd(1+pp)$; at maximi minimive formula eſt $\int yydx$. Quærantur jam harum trium formularum valores differentiales. Ac primo quidem erit formulæ $\int ydx$ valor differentialis $=nv.dx$; deinde formulæ

mulæ $\int dx\sqrt{(1+pp)}$ valor differentialis eſt $= - m.d. \frac{p}{\sqrt{(1+pp)}}$: & tertio formulæ $\int yy\,dx$ valor differentialis eſt $= 2 n\nu . y\,dx$. Ex quibus tribus valoribus differentialibus conficietur pro curva quæſita iſta æquatio, $0 = \alpha\,dx - \beta d. \frac{p}{\sqrt{(1+pp)}} + 2\gamma y\,dx$, ſeu $ccd. \frac{p}{\sqrt{(1+pp)}} = b\,dx + 2y\,dx = \frac{cc\,dp}{(1+pp)^{3:2}}$. Multiplicetur hæc æquatio per p & integretur; habebitur $ff + by + yy = \frac{-cc}{\sqrt{(1+pp)}}$, ubi tam cc quam ff pro arbitrio ſive affirmative ſive negative accipere licet. Hinc porro fiet $(ff+by+yy)^2 (1+pp) = c^4$; & $p = \frac{\sqrt{(c^4 - (ff+by+yy)^2)}}{ff+by+yy} = \frac{dy}{dx}$; ideoque $dx = \frac{(ff+by+yy)\,dy}{\sqrt{(c^4-(ff+by+yy)^2)}}$; quæ eſt æquatio pro curva Elaſtica. Ingredietur autem, per integrationem unam reliquam, nova quarta conſtans arbitraria; atque hiſce quatuor conſtantibus effici poterit ut curva per data duo puncta tranſeat; deinde binis reliquis conſtantibus obtinebitur, ut poſito $x = a$, tam area curvæ quam ejus longitudo datam magnitudinem conſequantur. Inſuper autem ambiguitate ſignorum, qua ſignum radicale afficitur, alterum ſignum præbebit curvam maximi, alterum minimi proprietate gaudentem. Quoniam autem in æquatione inventa data illa abſciſſæ magnitudo a non ineſt; ſequitur curvæ inventæ portionem quamvis cuicunque abſciſſæ reſpondentem hac quoque gaudere prærogativa ut, inter omnes alias curvas eidem illi abſciſſæ reſpondentes, & per eadem duo puncta tranſeuntes, quæ ſimul cum illa curva tum æqualem longitudinem quam æqualem aream complectantur, ut illa, inquam, curva circa abſciſſam ſuam rotata generet ſolidum maximæ minimæve capacitatis. Duo ſcilicet puncta, per quæ curva quæſita tranſeat, ideo hîc in conſiderationem ſunt ducenda, quia calculus præbuit æquationem differentialem ſecundi gradus, quæ per ſe duplicem determinationem requirit. Poterunt vero etiam

binæ

Digitized by Google

binæ reliquæ conſtantes, quæ ſtatim in æquatione inventa inerant, per puncta determinari, hocque pacto determinata ſolutio hujuſmodi emerget, quæ docebit per quatuor data puncta curvam deſcribere, quæ inter omnes alias per eadem quatuor puncta tranſeuntes, atque cum æque longas tum æquales areas continentes, producat circa axem rotata ſolidum vel maximum vel minimum. Perpetuo nimirum numerus conſtantium arbitrariarum, quæ in æquatione inventa cum actu tum potentia inſunt, declarabit quot determinationes ſint adhibendæ ut curva determinetur; hæcque deinde, inter omnes alias curvas iiſdem determinationibus præditas, quæſito ſatisfaciet.

EXEMPLUM II.

23. *Inter omnes lineas eidem abſciſſæ reſpondentes, quæ primo æquales contineant areas* $\int y\,dx$, *atque præterea circum axem rotatæ æqualia generent ſolida* $\int yy\,dx$; *determinare eam quæ ſuum gravitatis centrum vel maxime vel minime habeat elevatum: hoc eſt in qua ſit* $\frac{\int yx\,dx}{\int y\,dx}$ *vel maximum vel minimum.*

Sit abſciſſæ longitudo præſcripta, ad quam Solutionem accommodari oportet, $= a$; atque pro hac abſciſſa fiat valor formulæ $\int y\,dx = A$, formulæ $\int yy\,dx = B$, & formulæ $\int yx\,dx = C$. Porro ſit valor differentialis formulæ $\int y\,dx = dA = dx$; formulæ $\int yy\,dx = dB = 2y\,dx$, & formulæ $\int yx\,dx = dC = x\,dx$; ſumptis nimirum harum formularum valoribus differentialibus ſecundum regulas ſupra datas: omittendo tantum particulam $n\nu$, quippe quæ perpetuo per diviſionem tollitur. Cum jam maximi minimive expreſſio ſit, non ſimplex formula, ſed fractio $\frac{\int yx\,dx}{\int y\,dx}$; ejus valor differentialis erit $= \frac{A\,dC - C\,dA}{A^2}$ $= \frac{Ax\,dx - C\,dx}{A^2}$; atque, ob proprietatum binarum communium $\int y\,dx$ & $\int yy\,dx$ valores differentiales datos, nempe $dA = dx$ & $dB = 2y\,dx$, reſultabit pro curva quæſita ſequens æquatio

tio $\alpha dx + 2\,\mathfrak{C} y dx + \frac{\gamma A x dx - \gamma C dx}{A^2} = 0$, vel $(\alpha A^2 - \gamma C) dx + 2\,\mathfrak{C} A^2 y dx + \gamma A x dx = 0$; in qua æquatione, cum $\alpha, \mathfrak{C}, \gamma$ ſint conſtantes arbitrariæ, earum transformatione ſimul conſtantes determinatæ A & C ex computo expelli poſſunt; ita ut Solutio inventa ad omnes abſciſſas æque fiat accommodata. Pervenietur autem ad hanc æquationem $b dx = m y dx + n x dx$, ſeu, per dx diviſione inſtituta, $b = my + nx$, quæ æquatio eſt pro linea recta quacunque. Linea recta igitur ad axem verticalem utcunque ſita, inter omnes alias lineas cum axe tam eandem aream $\int y dx$ quam idem volumen $\int yy dx$ continentes, habebit ſuæ areæ centrum gravitatis vel maxime vel minime elevatum. Erit autem centrum gravitatis minime elevatum, ſi linea recta ſurſum cum axe convergat; maxime autem erit elevatum, ſi deorſum cum axe convergat; hique ſunt ambo caſus, quibus vel maxima vel minima centri gravitatis elevatio locum habet. Inter hos caſus eſt medius, quo linea illa recta ſit axi parallela: de quo dubium ſupereſſe poteſt, utrum centrum gravitatis ſit vel maxime depreſſum vel maxime elevatum. Verum iſte caſus nequidem in Quæſtione locum invenit. Nam poſita linea recta axi parallela, ita ut ſit $y = b$, tum omnino nulla alia exhiberi poteſt linea quæ pro eadem abſciſſa, cum æqualem aream $\int y dx$ tum æquale volumen $\int yy dx$ contineat: hocque ideo evenit, quod iſta linea recta, inter omnes alias lineas eandem aream $\int y dx$ comprehendentes, minimum volumen $\int yy dx$ includat.

EXEMPLUM III.

24. *Inter omnes curvas ejuſdem longitudinis* DAD *puncta data* D, D *jungentes, determinare eam cujus hæc ſit proprietas ut, ſi inter rectas verticales* DB, DB *per horizontalem* NN *ſpatium* NDADN *datæ magnitudinis abſcindatur, hujus ſpatii* NDADN *Centrum gravitatis imum obtineat locum.* *Fig.* 21.

Quæſtionis hujus Solutio eximium habet uſum in Hydroſtati-

ca, ejuſque ope ſolvetur Problema quo figura lintei D A D vaſi B D D B in punctis D D annexi inveſtigatur, quam induit, ſi vaſi data aquæ copia infundatur. Primo enim dum linteum extenſionem non admittit, longitudo curvæ D A D erit data, deinde etiam ſpatium N D A D N, quo quantitas aquæ infuſæ menſuratur, erit data: ac tertio, ſecundum genera'es Hydroſtaticæ & gravitationis leges, figuram D A D ita comparatam eſſe oportet, ut ſpatii N D A D N centrum gravitatis infimum occupet locum. Ad hoc Problema reſolvendum, ponatur D C $=$ C D $= a$, & ducta horizontali quacunque M P M, ſit M P $=$ P M $= x$, & A P $= y$, erit arcus M A M $=$ $2\int dx\sqrt{(1+pp)}$, poſito $dy = p\,dx$. Quod ſi jam longitudo curvæ D A D ponatur $= 2b$; æquatio inter x & y ita debet eſſe comparata, ut formula integralis $\int dx\sqrt{(1+pp)}$ fiat $= b$, poſito $x = a$. Porro area M A M ſit $= 2\int x\,dy = 2\int xp\,dx$; quæ fiat $= 2ff$, caſu quo ponitur $x = a$; ita ut tum ſit $\int xp\,dx = ff$. Hæc vero area non ipſa eſt data, ſed ea cum area N D D N datum ſpatium producere debet quod ſit $= 2cc$. Si igitur ponatur D N $= z$, erit $az + ff = cc$, & $z = \frac{cc - ff}{a} = \frac{cc - \int xp\,dx}{a}$, poſito $x = a$. Denique centrum gravitatis totius ſpatii N D A D N a puncto A diſtabit intervallo $= \frac{\int xyp\,dx + az(\mathrm{AC} + \frac{1}{2}z)}{cc}$, poſito poſt integrationem $x = a$; infra punctum C igitur centrum gravitatis ſitum erit intervallo $= \frac{\mathrm{AC}(cc - az) - \frac{1}{2}azz - \int xyp\,dx}{cc}$, quod debet eſſe maximum. Cum vero ſit $z = \frac{cc - \int xp\,dx}{a}$; maximum eſſe debebit hæc forma $\mathrm{AC}\int xp\,dx - \frac{c^4}{2a} + \frac{cc\int xp\,dx}{a} - \frac{(\int xp\,dx)^2}{2a}$ $- \int xyp\,dx$. Problema itaque huc redit ut, inter omnes curvas ejuſdem longitudinis datæ abſciſſæ D C $= a$ reſpondentes, definiatur ea in qua ſit hæc expreſſio $h\int xp\,dx + \frac{cc}{a}\int xp\,dx -$

$\frac{1}{2a}$

$\frac{1}{2a}(\int xpdx)^2 - \int xypdx$ maximum, exiſtente $y = b$, poſito $x = a$. Jam quia longitudo curvæ eſt $= \int dx\sqrt{(1+pp)}$, erit ejus valor differentialis $= -d.\frac{p}{\sqrt{(1+pp)}}$. Deinde formulæ $\int xpdx$ valor differentialis eſt $= -dx$, & valor differentialis formulæ $\int xypdx = xpdx - d.xy = -ydx$. Hinc totius expreſſionis, quæ maximum eſſe debet, valor differentialis prodit $= -bdx - \frac{cc}{a}dx + \frac{ff}{a}dx + ydx$, quæ, ob b & ff conſtantes non determinatas, tranſit in hanc $kdx + ydx$; ubi k eſt conſtans arbitraria. Quocirca prodibit iſta æquatio pro curva quæſita $kdx + ydx = -gg\,d.\frac{p}{\sqrt{(1+pp)}}$; quæ, per p multiplicata & integrata, dabit $m + 2ky + yy = \frac{2gg}{\sqrt{(1+pp)}}$; quam curvam conſtat eſſe Elaſticam, manebitque ea invariata, quemcunque valorem obtineat quantitas cc. Quæſtioni ergo propoſitæ ita ſatisfiet, ut per data puncta D & D curva Elaſtica traducatur, cujus axis ſeu diameter orthogonalis ſit recta verticalis AC, & cujus portio DAD datam obtineat longitudinem $2b$; hocque pacto, Solutio omnino erit determinata, unicaque curva ſatisfaciens reſultabit. Quod autem quantitas ſpatii NDADN $= 2cc$, de cujus centro gravitatis quæſtio eſt, prorſus ex computo exceſſerit, id quidem facile prævidere licuiſſet; quo pacto, Solutio multo facilior extitiſſet. Verum data opera hanc conditionem, etſi inutilem, adjecimus, ut modus pateret alia iſtiuſmodi Problemata, ubi talis reductio locum non invenit, reſolvendi.

SCHOLION II.

25. Sic igitur expoſita eſt univerſa Methodus maximorum & minimorum indeterminata, qua linea curva quæri ſolet maximi minimive proprietate quapiam prædita. Itaque Methodus tota perducta eſt ad inventionem valorum differentialium, qui ex unius tantum applicatæ incremento oriuntur. Scilicet ſi Pro-

blema poſtulet, inter omnes omnino curvas ad eandem abſciſſam relatas, eam in qua expreſſio quæpiam indefinita maximum minimumve obtineat valorem; tum illius expreſſionis quærendus eſt valor differentialis; qui nihilo æqualis poſitus dabit æquationem pro Curva quæſita. Quod ſi autem, inter omnes curvas quæ una pluribusve proprietatibus communibus gaudeant, eam definiri oporteat, in qua valor cujuſpiam expreſſionis propoſitæ fiat maximus vel minimus; tum, tam ſingularum proprietatum communium, quam maximi minimive, expreſſionis quæri debent valores differentiales, hique ſinguli per conſtantes arbitrarias multiplicari, quorum productorum ſumma nihilo æqualis poſita dabit æquationem pro Curva quæſita. Ad valorem autem differentialem cujuſque expreſſionis indeterminatæ inveniendum, Regulas in ſuperioribus Capitibus ſufficientes atque admodum faciles tradidimus. Ejuſmodi enim expreſſio indeterminata, ſive proprietatem communem continens, ſive maximum minimumve, perpetuo vel eſt formula integralis ſimplex, vel functio duarum pluriumve hujuſmodi formularum integralium. Quod vero ad formulas integrales ſimplices attinet; in Cap. IV §. 7 præcepta expoſuimus, quorum ope ejuſmodi formularum valores differentiales reperiri queant; ubi hanc indagationem ad quinque caſus reduximus. Quemadmodum autem ſecundum hæc eadem præcepta, cujuſcunque functionis duarum pluriumve formularum integralium ſimplicium valor differentialis conveniens definiri queat, id in ejuſdem Cap. IV, Propoſitione 4, indicavimus, modumque differentiationis ſimilem atque ſatis facilem expoſuimus: ita ut in hoc genere nihil ſupereſſe videatur, quod inſuper ſit adjiciendum.

FINIS.

A D-

Digitized by Google

Tabula. II.

Fig. 12.

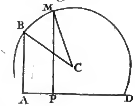


Fig. 13.

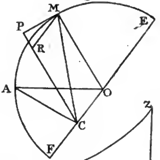


Fig. 15.

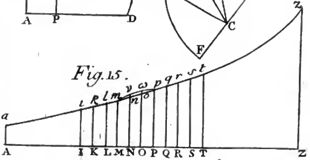


Fig. 17.

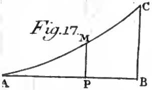


Fig. 18.

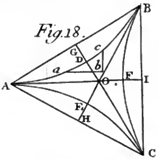


Fig. 20.

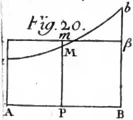
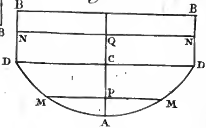




Fig. 21.



Digitized by Google

Digitized by Google

ADDITAMENTUM I.

De Curvis Elaſticis.

I.

JAm pridem ſummi quique Geometræ agnoverunt, Methodi in hoc Libro traditæ non ſolum maximum eſſe uſum in ipſa Analyſi, ſed etiam eam ad reſolutionem Problematum phyſicorum ampliſſimum ſubſidium afferre. Cum enim Mundi univerſi fabrica ſit perfectiſſima, atque a Creatore ſapientiſſimo abſoluta, nihil omnino in mundo contingit, in quo non maximi minimive ratio quæpiam eluceat: quamobrem dubium prorſus eſt nullum, quin omnes Mundi effectus ex cauſis finalibus, ope Methodi maximorum & minimorum æque feliciter determinari queant, atque ex ipſis cauſis efficientibus. Hujus rei vero paſſim tam eximia extant ſpecimina, ut ad veritatis confirmationem pluribus Exemplis omnino non indigeamus; quin potius in hoc erit elaborandum, ut, in quovis Quæſtionum naturalium genere, ea inveſtigetur quantitas, quæ maximum minimumve induat valorem: quod negotium ad Philoſophiam potius quam ad Matheſin pertinere videtur. Cum igitur duplex pateat via effectus Naturæ cognoſcendi; altera per cauſas efficientes, quæ Methodus directa vocari ſolet; altera cauſas finales; Mathematicus utrâque pari ſucceſſu utitur. Quando ſcilicet cauſæ efficientes nimis ſunt abſconditæ, finales autem noſtram cognitionem minus effugiunt; per Methodum indirectam Quæſtio ſolet reſolvi: e contrario autem Methodus directa adhibetur, quoties ex cauſis efficientibus effectum definire licet. Inprimis autem opera eſt adhibenda, ut per utramque viam aditus ad Solutionem aperiatur: ſic enim non ſolum altera Solutio per alteram maxime confirmatur, ſed etiam ex utriuſque conſenſu

Digitized by Google

ſummam percipimus voluptatem. Hoc modo, curvatura funis ſeu catenæ ſuſpenſæ duplici via eſt eruta; altera a priori, ex ſollicitationibus gravitatis; altera vero per Methodum maximorum ac minimorum, quoniam funis ejuſmodi curvaturam recipere debere intelligebatur, cujus centrum gravitatis infimum obtineret locum. Similiter curvatura radiorum per medium diaphanum variæ denſitatis tranſeuntium, tam a priori eſt determinata, quam etiam ex hoc principio, quod tempore breviſſimo ad datum locum pervenire debeant. Plurima autem alia ſimilia exempla a Viris Celeberrimis BERNOULLIIS, aliiſque, ſunt prolata, quibus tam Methodus ſolvendi a priori, quam cognitio cauſarum efficientium maxima accepit incrementa. Quanquam igitur, ob hæc tam multa ac præclara ſpecimina, dubium nullum relinquitur, quin in omnibus lineis curvis, quas Solutio Problematum phyſico-mathematicorum ſuppeditat, maximi minimive cujuſpiam indoles locum obtineat; tamen ſæpenumero hoc ipſum maximum vel minimum difficillime perſpicitur; etiamſi a priori Solutionem eruere licuiſſet. Sic etſi figura, quam lamina elaſtica incurvata induit, jam pridem eſt cognita; tamen quemadmodum ea curva per Methodum maximorum & minimorum, hoc eſt, per cauſas finales, inveſtigari poſſit a nemine adhuc eſt animadverſum. Quamobrem cum Vir Celeberrimus, atque in hoc ſublimi naturam ſcrutandi genere perſpicaciſſimus, *Daniel* BERNOULLI mihi indicaſſet ſe univerſam vim, quæ in lamina elaſtica incurvata inſit, una quadam formula quam *vim potentialem* appellat, complecti poſſe; hancque expreſſionem in curva Elaſtica minimam eſſe oportere; quoniam hoc invento Methodus mea maximorum ac minimorum hoc Libro tradita mirifice illuſtratur, ejuſque uſus ampliſſimus maxime evincitur; hanc occaſionem exoptatiſſimam prætermittere non poſſum, quin, hanc inſignem curvæ Elaſticæ proprietatem a Celeb. BERNOULLIO obſervatam publicando, ſimul Methodi meæ uſum clarius patefaciam. Continet enim iſta proprietas in ſe differentialia ſecundi gradus, ita ut ei evolvendæ Methodi Problema iſoperimetricum ſolvendi ante traditæ non ſufficiant.

2. Sit

2. Sit A B lamina Elaſtica utcunque incurvata; vocetur arcus $AM = s$, & radius oſculi curvæ $MR = R$: atque, ſecundum BERNOULLIUM, exprimetur *vis potentialis* in laminæ portione A M contenta hac formula $\int \frac{ds}{RR}$, ſiquidem lamina ſit ubique æqualiter craſſa, lata & elaſtica, atque in ſtatu naturali in directum extenſa. Hinc iſta erit curvæ A M indoles, ut in ea hæc expreſſio omnium minimum obtineat valorem. Quoniam vero in radio oſculi *R* differentialia ſecundi gradus inſunt, ad curvam hac proprietate præditam determinandam quatuor opus erit conditionibus, id quod cum Quæſtionis natura apprime convenit. Cum enim per datos terminos A & B infinitæ laminæ Elaſticæ eæque ejuſdem longitudinis inflecti queant, quæſtio non erit determinata, niſi præter duo puncta A & B, ſimul alia duo puncta, ſeu quod eodem redit poſitio tangentium in punctis extremis A & B præſcribatur. Propoſita namque lamina Elaſtica, longiori quam eſt diſtantia punctorum A & B; ea non ſolum ita incurvari poteſt, ut intra terminos A & B contineatur, ſed etiam ut ejus tangentes in punctis hiſce datas teneant directiones. His notatis; Quæſtio de invenienda curvatura laminæ Elaſticæ, ex hoc fonte reſolvenda, ita debet proponi: *ut, inter omnes curvas ejuſdem longitudinis, quæ non ſolum per puncta* A *&* B *tranſeant, ſed etiam in his punctis a rectis poſitione datis tangantur, definiatur ea in qua ſit valor hujus expreſſionis* $\int \frac{ds}{RR}$ *minimus.* *De curvatura Laminæ Elaſticæ uniformis. Fig. 1.*

3. Quia ſolutionem ad coordinatas orthogonales accommodari convenit, ſumatur recta quæcunque A D pro axe, in qua ſit abſciſſa $AP = x$, applicata $PM = y$; ponatur, uti Methodus tradita jubet, $dy = p\,dx$, $dp = q\,dx$; erit elementum curvæ $Mm = ds = dx\sqrt{(1 + pp)}$. Primum ergo quia curvæ, ex quibus quæſita erui debet, iſoperimetræ ſtatuuntur, habebitur iſta expreſſio conſideranda $\int dx\sqrt{(1 + pp)}$; quæ cum generali $\int Z dx$ comparata hunc præbet valorem differentialem *Fig. 2.*

Digitized by Google

$\frac{1}{dx}\, d.\, \frac{p}{\sqrt{(1+pp)}}$. Deinde cum ſit radius oſculi $= \frac{dx(1+pp)^{3:2}}{dp}$ $= \frac{(1+pp)^{3:2}}{q} = R$, formula $\int \frac{ds}{RR}$, quæ minimum eſſe debet, abit in $\int \frac{qq\,dx}{(1+pp)^{5:2}}$. Comparetur hæc cum forma generali $\int Z\,dx$; erit $Z = \frac{qq}{(1+pp)^{5:2}}$, & poſito $dZ = M\,dx + N\,dy + P\,dp + Q\,dq$, erit $M = 0$, $N = 0$, $P = \frac{-5pqq}{(1+pp)^{7:2}}$, & $Q = \frac{2q}{(1+pp)^{5:2}}$. Valor ergo differentialis ex hac formula $\int \frac{qq\,dx}{(1+pp)^{5:2}}$ oriundus, erit $-\frac{dP}{dx} + \frac{ddQ}{dx^2}$. Quamobrem pro curva quæſita hæc habebitur æquatio, $\frac{\alpha}{dx}\, d.\, \frac{p}{\sqrt{(1+pp)}}$ $= \frac{dP}{dx} - \frac{ddQ}{dx^2}$; quæ, per dx multiplicata & integrata, dat $\frac{\alpha p}{\sqrt{(1+pp)}} + \beta = P - \frac{dQ}{dx}$. Multiplicetur hæc æquatio per $q\,dx = dp$, ut prodeat $\frac{\alpha p\,dp}{\sqrt{(1+pp)}} + \beta dp = P\,dp - q\,dQ$. Cum autem, ob $M = 0$ & $N = 0$, ſit $dZ = P\,dp + Q\,dq$, erit $P\,dp = dZ - Q\,dq$, quo valore loco $P\,dp$ ſubſtituto, emerget $\frac{\alpha p\,dp}{\sqrt{(1+pp)}} + \beta dp = dZ - Q\,dq - q\,dQ$; quæ denuo integrata dat $\alpha\sqrt{(1+pp)} + \beta p + \gamma = Z - Qq$. Jam cum ſit $Z = \frac{qq}{(1+pp)^{5:2}}$, & $Q = \frac{2q}{(1+pp)^{5:2}}$, erit $\alpha\sqrt{(1+pp)} + \beta p + \gamma = \frac{-qq}{(1+pp)^{5:2}}$. Sumantur conſtantes arbitrariæ α, β, & γ negative, eritque $q = (1+pp)^{5:4}$
$\times\sqrt{(\alpha}$

$\times \sqrt{(\alpha\sqrt{(1+pp)}+\beta p+\gamma)} = \frac{dp}{dx}$. Hinc ergo elicitur ſequens æquatio

$$dx = \frac{dp}{(1+pp)^{5:4}\sqrt{(\alpha\sqrt{(1+pp)}+\beta p+\gamma)}}.$$

Deinde ob $dy = pdx$, habebitur quoque

$$dy = \frac{pdp}{(1+pp)^{5:4}\sqrt{(\alpha\sqrt{(1+pp)}+\beta p+\gamma)}};$$

quæ duæ æquationes ſufficerent ad curvam per quadraturas conſtruendam.

4. Harum formularum ſic in genere ſpectatarum neutra eſt integrabilis; combinari autem certo quodam modo poſſunt, ut aggregatum integrationem admittat. Cum enim ſit

$$d.\frac{2\sqrt{(\alpha\sqrt{(1+pp)}+\beta p+\gamma)}}{\sqrt{\sqrt{(1+pp)}}} = \frac{dp(\beta-\gamma p)}{(1+pp)^{5:4}\sqrt{(\alpha\sqrt{(1+pp)}+\beta p+\gamma)}}$$

erit $\frac{2\sqrt{(\alpha\sqrt{(1+pp)}+\beta p+\gamma)}}{(1+pp)^{1:4}} = \beta x - \gamma y + \delta$. Quoniam axis poſitio eſt arbitraria, conſtans δ ſine defectu amplitudinis omitti poteſt. Deinde vero etiam axis ita mutari poteſt ut fiat $\frac{\beta x - \gamma y}{\sqrt{(\beta\beta+\gamma\gamma)}}$ abſciſſa, eritque applicata $\frac{\gamma x + \beta y}{\sqrt{(\beta\beta+\gamma\gamma)}}$; hinc etiam tuto γ nihilo æqualis poni poteſt, quia nihil impedit, quominus illa nova abſciſſa per x exprimatur. Hanc ob rem; habebimus pro curva Elaſtica iſtam æquationem

$2\sqrt{(\alpha\sqrt{(1+pp)}+\beta p)} = \beta x(1+pp)^{1:4}$; quæ, ſumptis quadratis, dat $4\alpha\sqrt{(1+pp)}+4\beta p = \beta^2 x^2\sqrt{(1+pp)}$. Sit, ad homogeneitatem introducendam, $\alpha = \frac{4m}{aa}$ & $\beta = \frac{4n}{aa}$ erit $naap = (nnxx - maa)\sqrt{(1+pp)}$, unde $n^2a^4pp = (nnxx - maa)^2(1+pp)$; ideoque $p = \ldots$

$\frac{nnxx - maa}{\sqrt{(n^2a^4 - (nnxx - maa)^2)}} = \frac{dy}{dx}$. Mutatis ergo conſtantibus, atque abſciſſam x data conſtante ſive augendo ſive minuendo; habebitur hujuſmodi æquatio pro curva Elaſtica generalis:

$dy = \frac{(\alpha + \beta x + \gamma xx)\,dx}{\surd(a^4 - (\alpha + \beta x + \gamma xx)^2)}$, ex qua oritur
$ds = \frac{aadx}{\surd(a^4 - (\alpha + \beta x + \gamma xx)^2)}$; ex quibus æquationibus conſenſus hujus curvæ inventæ cum curva Elaſtica jam pridem eruta manifeſto elucet.

5. Quo autem iſte conſenſus clarius ob oculos ponatur, naturam curvæ Elaſticæ a priori quoque inveſtigabo; quod etſi jam a Viro ſummo *Jacobo* BERNOULLIO excellentiſſime eſt factum; tamen, hac idonea occaſione oblata, nonnulla circa indolem curvarum Elaſticarum, earumque varias ſpecies & figuras adjiciam; quæ ab aliis vel prætermiſſa, vel leviter tantum pertractata eſſe video.

Fig. 3. Sit lamina Elaſtica AB in B ita muro ſeu pavimento firmo infixa, ut hæc extremitas B non ſolum firmiter retineatur, ſed etiam tangentis in B poſitio determinetur. In A autem lamina connexam habeat virgam rigidam AC, cui normaliter applicata ſit vis CD $= P$, qua lamina in ſtatum incurvatum BMA redigatur. Sumatur hæc recta AC producta pro axe, ac, poſita AC $= c$, ſit abſciſſa AP $= x$, applicata PM $= y$. Quod ſi jam lamina in M omnem elaſticitatem ſubito amitteret, ac perfecte flexilis evaderet; a vi P utique inflecteretur, inflexione proficiſcente a vis P momento $= P(c + x)$. Quominus ergo hæc inflexio actu ſequatur, elaſticitas laminæ in M in æquilibrio conſiſtit cum vis ſollicitantis momento $P(c + x)$. Elaſticitas autem primo ab indole materiæ ex qua lamina conſtat, & quam ubique eandem ſtatuo, pendet; tum vero ſimul ab incurvatione laminæ in puncto M, ita ut ſit reciproce proportionalis radio oſculi in M. Sit ergo radius oſculi in M $= R = \frac{ds^3}{-dxddy}$; exiſtente $ds = \surd(dx^2 + dy^2)$ & dx conſtante; atque exprimat $\frac{Ekk}{R}$ vim Elaſticam laminæ in M, quæ cum momento vis ſollicitantis $P(c + x)$ in æquilibrio conſiſtat, ita ut ſit $P(c + x) = \frac{Ekk}{R} = \frac{-Ekkdxddy}{ds^3}$. Æquatio hæc

per

per dx multiplicata fit integrabilis, eritque integrale

$$P\,(xx + cx + f) = \frac{-Ekkdy}{\surd(dx^2 + dy^2)};$$ unde oritur

$$dy = \frac{-Pdx\,(\tfrac{1}{2}xx + cx + f)}{\surd(E^2k^4 - P^2(\tfrac{1}{2}xx + cx + f)^2)},$$ quæ æquatio omnino convenit cum ea, quam modo per Methodum maximorum ac minimorum ex principio *Bernoulliano* elicui.

6. Ex comparatione hujus æquationis cum ante inventa, definiri poterit vis quæ requiritur ad datam laminæ curvaturam inducendam; ſiquidem curvatura contineatur in æquatione generali inventa. Teneat ſcilicet lamina elaſtica figuram AMB, cujus natura exprimatur hac æquatione

$$dy = \frac{(\alpha + \beta x + \gamma xx)\,dx}{\surd(a^4 - (\alpha + \beta x + \gamma xx)^2)};$$ exprimat vero Ekk hujus laminæ elaſticitatem abſolutam, ita ſcilicet, ut Ekk, in quovis loco, per radium oſculi diviſa præbeat vim elaſticam veram. Ad comparationem inſtituendam multiplicetur numerator & denominator per $\frac{Ekk}{aa}$, ut habeatur

$$dy = \frac{Ekkdx(\alpha + \beta x + \gamma xx) : aa}{\surd(E^2k^4 - \frac{E^2k^4}{a^4}(\alpha + \beta x + \gamma xx)^2)}.$$ Nunc ergo erit $-\tfrac{1}{2}P = \frac{Ekk\gamma}{aa}$; $-Pc = \frac{Ekk\beta}{aa}$; $-Pf = \frac{Ekk\alpha}{aa}$; ideoque vis CD ſollicitans $= \frac{-2Ekk\gamma}{aa}$; intervallum AC $= c = \frac{\beta}{2\gamma}$, & conſtans $f = \frac{\alpha}{2\gamma}$.

7. Ut igitur lamina elaſtica AB altero termino B muro infixa incurvetur in figuram AMB, cujus natura exprimitur hac æquatione $dy = \frac{(\alpha + \beta x + \gamma xx)\,dx}{\surd(a^4 - (\alpha + \beta x + \gamma xx)^2)}$, neceſſe eſt ut hæc lamina ſollicitetur in directione CD normali ad axem AP, ſumpta diſtantia AC $= \frac{\beta}{2\gamma}$, a vi CD $= \frac{-2Fkk\gamma}{aa}$; quæ vis ſcilicet in plagam contrariam, ac figura indicat, dirigetur, ſi

ſi γ fuerit quantitas poſitiva. Quia $\frac{Ekk}{R}$ æquivalet momento vis ſollicitantis, expreſſio $\frac{Ekk}{aa}$ homogenea erit ponderi ſeu vi puræ; quæ vis propterea $\frac{Ekk}{aa}$ cognoſcetur ex elaſticitate laminæ. Sit hæc vis $= F$; atque erit vis flectens CD ad hanc vim F ut -2γ ad 1; erit enim γ numerus purus.

8. Hinc porro definiri poteſt vis ad laminæ portionem BM in ſtatu ſuo conſervandam requiſita, ſi portio AM prorſus reſcindatur. Reſciſſa hac portione AM, deſinat lamina Elaſtica in virgam rigidam MT omnis flexionis expertem, quæ autem cum lamina ita ſit connexa, ut perpetuo tangentem in puncto M referat, utcunque lamina inclinetur. Hoc poſito, ex antecedentibus manifeſtum eſt, ad conſervationem curvaturæ BM requiri ut virga MT in puncto N trahatur in directione ND vi quæ ſit $= \frac{-2Ekk\gamma}{aa}$; directio autem ND erit normalis ad axem AP, atque intervallum AC erit $= \frac{\mathsf{6}}{2\gamma}$. Diſtantia itaque MN fiet $= \frac{ds}{dx} CP = \frac{ds}{dx} \cdot \frac{\mathsf{6} + 2\gamma x}{2\gamma} = \frac{(\mathsf{6} + 2\gamma x) ds}{2\gamma dx}$ eſt vero $\frac{ds}{dx} = \frac{aa}{\surd(a^4 - (\alpha + \mathsf{6}x + \gamma xx)^2)}$. Quod ſi hæc vis $ND = \frac{-2Ekk\gamma}{aa}$ reſolvatur in normalem NQ ad tangentem MT, & tangentialem NT, erit vis normalis $NQ = \frac{-2Ekk\gamma}{aa} \cdot \frac{dx}{ds}$, & vis tangentialis $NT = \frac{-2Ekk\gamma}{aa} \cdot \frac{dy}{ds}$.

9. Sin autem pars BM reſcindatur, relicta parte AM, quæ in directione CD ſollicitatur ut ante vi $= \frac{-2Ekk\gamma}{aa}$; ad curvaturam AM conſervandam extremitas M, quæ connexa intelligatur cum virga rigida tangente MN, ſollicitari debebit in puncto N a vi pariter $= \frac{-2Ekk\gamma}{aa}$, ſed in directione contraria

traria ei, quam caſu præcedente invenimus. Perpetuo enim vires utrique extremitati laminæ incurvatæ aplicandæ ſe mutuo deſtruere, atque adeo æquales & directiones oppoſitas habere debent. Alioquin enim tota lamina moveretur; ad quem motum compeſcendum opus foret vi æquilibrium inter vires ſollicitantes producente. Hinc ergo vires cuicunque portioni laminæ reſectæ applicandæ facillime definiri poſſunt, quæ jam inductam curvaturam conſervent.

10. Sit AM lamina Elaſtica incurvata, quæ in A & M annexas habeat virgas rigidas AD, MN, quibus in directionibus *Fig. 4.* directe oppoſitis DE, NR applicatæ ſint vires æquales DE, NR, quæ in æquilibrio conſiſtentes laminæ curvaturam AM inducant, pro qua æquationem quæri oporteat. Primum ergo, pro axe ſumatur recta AP per punctum A tranſiens, atque ad directionem vis ſollicitantis ER normalis. Ponatur Elaſticitas laminæ abſoluta $= Ekk$: ſitque anguli CAD, quem tangens AD in A cum axe conſtituit, & qui eſt datus, ſinus $= m$, coſinus $= n$, exiſtente ſinu toto $= 1$, ita ut ſit $mm + nn = 1$. Vocetur porro diſtantia $AC = c$, & vis flectens $DE = NR = P$; ac, poſitis abſciſſa $AP = x$, applicata $PM = y$, natura curvæ hac exprimetur æquatione

$$dy = \frac{-P\,dx\,(\frac{1}{2}xx + cx + f)}{\sqrt{(E^2k^4 - P^2(\frac{1}{2}xx + cx + f)^2)}}.$$

Quoniam vero directio tangentis in A datur, poſito $x = 0$, fieri debet $\frac{dy}{dx} = \frac{m}{n}$; hinc ergo obtinebitur $\frac{m}{n} = \frac{-Pf}{\sqrt{(E^2k^4 - P^2f)}} = \frac{m}{\sqrt{(1 - mm)}}$, & $m = \frac{-Pf}{Ekk}$. Determinatur ergo hinc conſtans f, ita ut ſit $f = \frac{-mEkk}{P}$, ideoque hinc tota curva determinatur.

11. Ad curvaturam ergo ſuperiori æquatione expreſſam laminæ AM inducendam, tangenti AD in puncto D, ita ut *Fig. 5.* ſit $AD = \frac{c}{n}$, applicatam eſſe oportet vim $DE = P$; cujus

directio ſit parallela applicatis P M. Reſolvatur hæc vis D E in duas laterales D d, D f, inter ſe normales; erit vis D d $= Pn$ & vis D f $= Pm$. Quo jam conſideratio rectæ A D ex computo expellatur, loco vis D d, in datis punctis A & B, ſumpto intervallo A B $= b$, duæ vires ſubſtitui poſſunt, A a $= p$; B b $= q$; normales pariter ad virgam A B, ſumendo $pb = Pn$. B D $= nP\left(\frac{c}{n} - b\right)$, & $q = p + nP$. Quia deinceps perinde eſt, in quonam virgæ A D puncto applicetur vis tangentialis D f $= mP$, applicetur ea in ipſo puncto A ponendo A F $= nP$. Sit autem hæc vis A F $= r$, ita ut lamina M A a tribus viribus A a $= p$, B b $= q$, & A F $= r$ ſollicitetur, a quibus, qualis incurvatio oriatur, inveſtigemus.

12. Primo ergo, cum ſit $mP = r$, erit $P = \frac{r}{m}$, qui valor ſubſtitutus in prioribus æquationibus dabit $pb = \frac{cr}{m} - \frac{nbr}{m}$, & $q = p + \frac{nr}{m}$. Hinc erit $\frac{n}{m} = \frac{q-p}{r}$; ex qua æquatione primum poſitio axis A P innoteſcit; erit nempe tangens anguli C A D $= \frac{r}{q-p}$: hinc $m = \frac{r}{\surd(r^2+(q-p)^2)}$ & $n = \frac{q-p}{\surd(r^2+(q-p)^2)}$. Deinde ex æquatione $bp = \frac{cr}{m} - \frac{nbr}{m} = \frac{cr}{m} - bq + bp$; fit $c = \frac{mbq}{r}$, ſeu $c = \frac{bq}{\surd(r^2+(q-p)^2)}$; atque $P = \surd(rr+(q-p)^2)$. Cum autem ſit $f = \frac{-mEkk}{P} = \frac{-Ekkr}{rr+(q-p)^2}$ erit $\frac{1}{2}xx + cx + f = \frac{1}{2}xx + \frac{bqx}{\surd(r^2+(q-p)^2} - \frac{Ekkr}{rr+(q-p)^2}$; unde pro curva quæſita iſta obtinebitur æquatio

$$dy = \frac{dx\left(\frac{Ekkr}{\surd(rr+(q-p)^2)} - bqx - \frac{1}{2}xx\surd(rr+(q-p)^2)\right)}{\surd\left(E^2k^4 - \left(\frac{Ekkr}{\surd(rr+(q-p)^2)} - bqx - \frac{1}{2}xx\surd(rr+(q-p)^2)\right)^2\right)}.$$

Hæc autem æquatio maxime eſt accommodata ad modum maxime

xime conſuetum laminas incurvandi, dum eæ vel forcipe vel duobus digitis apprehenduntur; quorum alter laminam in directione Aa, alter in directione Bb urget, præter quas vires lamina inſuper in directione AF protrahi poteſt.

13. Si vis tangentialis $AF = r$ evaneſcat; incidet axis AP in ipſam tangentem AF, productam, eritque tum

$$dy = \frac{-dx(bqx + \frac{1}{2}(q-p)xx)}{\sqrt{(E^2k^4 - (bqx + \frac{1}{2}(q-p)xx)^2)}}.$$

Sin autem vires normales p & q fiant inter ſe æquales; erit axis AP normalis ad tangentem AF, ob $n = 0$; & pro curva orietur hæc æquatio

$$dy = \frac{dx(Ekk - bqx - \frac{1}{2}rxx)}{\sqrt{(2Ekk(bqx + \frac{1}{2}rxx) - (bqx - \frac{1}{2}rxx)^2)}}.$$

Hic ſi præterea fuerit $r = 0$, ita ut lamina in punctis A & B urgeatur a viribus æqualibus Aa, Bb, contrariis tantum, natura curvæ exprimetur hac æquatione $dy = \frac{dx(Ekk - bqx)}{\sqrt{bq(2Ekkx - bqxx)}}$, quæ integrata dat $y = \sqrt{\frac{2Ekkx - bqxx}{bq}}$; quæ eſt pro Circulo, lamina ergo hoc caſu in arcum Circuli incurvatur, cujus radius erit $= \frac{Ekk}{bq}$.

14. Cum igitur videamus non ſolum Circulum in curvarum Elaſticarum claſſe contineri, ſed etiam in ipſis infinitam varietatem locum habere; operæ pretium erit hic enumerationem omnium variarum ſpecierum in hoc curvarum genere contentarum inſtituere. Hoc enim modo non ſolum indoles harum curvarum penitius perſpicietur; ſed etiam, caſu quocunque oblato, ex ſola figura dijudicare licebit, ad quamnam ſpeciem curva formata referri debeat. Eodem autem modo hic ſpecierum diverſitatem conſtituemus, quo vulgo linearum algebraïcarum ſpecies, in dato ordine contentæ enumerari ſolent.

Enumeratio curvarum Elaſticarum.

15. Æquatio generalis pro curvis Elaſticis . . . .

$$dy = \frac{(\alpha + \beta x + \gamma xx)dx}{\sqrt{(a^4 - (\alpha + \beta x + \gamma xx)^2)}},$$ initio abſciſſarum in axe

per

per intervallum $\frac{6}{2\gamma}$ promoto, & pro $\frac{aa}{\gamma}$ ſcribendo aa, ſeu ponendo $\gamma = 1$, accipiet hanc formam ſimpliciorem: $dy = \frac{(\alpha + xx)\,dx}{\sqrt{(a^4 - (\alpha + xx)^2)}}$. Quia vero eſt $a^4 - (\alpha + xx)^2 = (aa - \alpha - xx)(aa + \alpha + xx)$; ponatur $aa - \alpha = cc$, ut ſit $\alpha = aa - cc$, atque æquatio tranſibit in hanc formam $dy = \frac{(aa - cc + xx)\,dx}{\sqrt{(cc - xx)(2aa - cc + xx)}}$. Qua æquatione exprimatur natura curvæ AMC, poſita abſciſſa AP $= x$, & applicata PM $= y$. Cum ergo ſit $6 = 0$, directio vis laminam Elaſticam incurvans erit ad axem AP in ipſo puncto A normalis, ideoque AD repræſentabit directionem vis ſollicitantis, quæ vis ipſa erit $= \frac{2Ekk}{aa}$, exprimente Ekk elaſticitatem abſolutam.

Fig. 6.

16. Si ponatur $x = 0$, erit $\frac{dy}{dx} = \frac{aa - cc}{c\sqrt{(2aa - cc)}}$; quæ expreſſio præbet tangentem anguli quem curva AM in A cum axe AP conſtituit; cujus anguli ſinus erit $= \frac{aa - cc}{aa}$. Quare ſi fuerit $aa = \infty$, lamina in puncto A erit normalis ad axem AP, nullamque habebit curvaturam, propterea quod vis incurvans $\frac{2Ekk}{aa}$ evaneſcit. Caſu ergo quo $a = \infty$, prodit laminæ figura naturalis, hoc eſt linea recta: quæ ergo primam ſpeciem linearum Elaſticarum conſtituit, quam repræſentabit recta AB utrimque in infinitum producta.

17. Antequam reliquas ſpecies enumeremus, conveniet in genere circa figuram Elaſticæ quaſdam obſervationes inſtituere. Intelligitur autem angulus PAM, quem curva in A cum axe AP conſtituit, decreſcere, quo minor evadat quantitas aa, hoc eſt quo magis vis incurvans $\frac{2Ekk}{aa}$ intendatur. Atque ſi evadat $aa = cc$, tum axis AP ipſe curvam in A tanget. Quod ſi autem fuerit $aa < cc$, tum curva AM, quæ adhuc deorſum excurrebat, nunc ſurſum verget, quoad fiat $aa = \frac{1}{2}cc$; quo caſu tangens

Digitized by Google

tangens curvæ in rectam Ab incidet. At ſi fiat $aa < \frac{1}{2}cc$, tum angulus PAM prorſus fiet imaginarius, ideoque in A nulla exiſtet curvæ portio, qui diverſi caſus ſpecierum varietatem conſtituent.

18. Ex æquatione porro intelligitur, quia formam ſuam non mutat, ſi coordinatæ x & y ambæ negativæ ſtatuantur, curvam circa A ramos habere ſimiles & æquales AMC & Amc alternatim diſpoſitos; ita ut in A ſit punctum flexus contrarii; unde cognita curvæ portione AMC, ſimul ejus continuatio Amc ultra A cognoſcetur, quippe quæ illi eſt ſimilis & æqualis. Sic ſumpta $Ap = AP$, erit quoque $pm = PM$. Recedendo autem ab A, curva utrimque magis ab axe reclinatur, donec ſumpta abſciſſa $= AE = c$, applicata EC curvam tangat; namque poſito $x = c$, fit $\frac{dy}{dx} = \infty$. Perſpicuum autem eſt abſciſſam x ultra $AE = c$ excreſcere non poſſe; alioquin enim fieret $\frac{dy}{dx}$ imaginarium; hinc ergo tota curva continebitur inter applicatas extremas EC & ec, ultra quos cancellos egredi non queat. Jam ergo generatim cognitos habemus binos curvæ ramos AC & Ac utrimque ab A uſque ad cancellos protenſos.

19. Videamus ergo quonam curſu curva ultra C & c progrediatur. Hunc in finem ſumamus rectam CD ipſi AE parallelam pro axe, ac ponamus has novas coordinatas $CQ = t$, $QM = u$; eritque $t + x = AE = CD = c$; & $y + u = CE = AD = b$; unde fit $x = c - t$ & $y = b - u$, ſeu $dy = -du$. His valoribus ſubſtitutis, orietur æquatio pro curva inter coordinatas $CQ = t$ & $QM = u$, quæ erit $du = \frac{(aa - 2ct + tt)dt}{\sqrt{t(2c - t)(2aa - 2ct + tt)}}$. Hic primum patet, ſi ſumatur t infinite parvum, fore $du = \frac{aadt}{2a\sqrt{ct}}$, ideoque $u = a\sqrt{\frac{t}{c}}$; quæ æquatio indicat curvam ultra C ſimili modo ver-

ſus N progredi incipere, quo ex C ad M extenditur. Ambiguitas autem ſigni $\surd$ in denominatore æquationis luculenter declarat, applicatam u æque negative accipi poſſe atque affirmative: unde manifeſtum eſt, rectam CD eſſe curvæ diametrum, atque adeo arcum CNB ſimilem & æqualem fore arcui CMA.

20. Simili autem modo recta cd, ex altera parte axi AE per c parallela ducta, erit curvæ diameter; propterea quod ramus Acb ſimilis & æqualis eſt ramo ACB. In punctis ergo B & b, erunt quoque puncta flexus contrarii omnino uti in A; unde curva ſimiliter ulterius progredietur. Habebit ergo curva infinitas diametros CD, cd, &c. intervallo eodem Dd a ſe invicem diſtantes ac parallelas inter ſe; hancque ob rem curva conſtabit ex infinitis partibus inter ſe ſimilibus & æqualibus; atque ideo tota curva cognoſcetur, ſi unica tantum portio AMC fuerit perſpecta.

21. Quia in A eſt punctum flexus contrarii, ibidem erit radius oſculi infinite magnus; id quod ex ipſa curvæ natura patet. Cum enim curva in A ſollicitetur a vi $= \frac{2Ekk}{aa}$ in directione AD; erit in quovis loco M, ſi radius oſculi ibi ponatur $= R$, ex natura elaſticitatis $\frac{2Ekk}{aa}x = \frac{Ekk}{R}$; unde fit $R = \frac{aa}{2x}$. In puncto ergo A radius oſculi eſt infinitus; at vero in punctis C, c, ob AE $=$ Ae $= c$, erit radius oſculi $= \frac{aa}{2c}$; in his ſcilicet locis maxime a recta BAb remotis curvatura eſt maxima.

22. Etſi autem pro puncto C conſtat abſciſſa AE $= c$ tamen diſtantia EC niſi per integrationem æquationis
$dy = \frac{(aa - cc + xx)\,dx}{\surd(cc - xx)(2aa - cc + xx)}$ definiri non poteſt. Si enim poſt integrationem ponatur $x = c$; valor ipſius y dabit diſtantiam CE, quæ bis ſumpta præbebit diſtantiam AB, ſeu intervallum Dd, inter diametros interjacens. Simili modo inte-

integratione opus erit ad laminæ incurvatæ AC longitudinem determinandam. Cum enim poſito arcu $AM = s$, ſit $ds = \frac{aa\,dx}{\sqrt{(cc - xx)(2aa - cc + xx)}}$, hujus integrale, poſito $x = c$, dabit longitudinem curvæ AC.

23. Cum autem iſtæ formulæ integrationem non admittant; per approximationem valores intervalli AD & arcus curvæ AC commode exprimere nitamur. Ponamus in hunc finem $\sqrt{(cc - xx)} = z$; eritque $PM = y = \int \frac{(aa - zz)\,dx}{z\sqrt{(2aa - zz)}}$, & $AM = s = \int \frac{aa\,dx}{z\sqrt{(2aa - zz)}}$. Eſt vero per ſeriem $\frac{1}{\sqrt{(2aa - zz)}}$

$$= \frac{1}{a\sqrt{2}}\left(1 + \frac{1}{4} \times \frac{zz}{aa} + \frac{1.3}{4.8} \times \frac{z^4}{a^4} + \frac{1.3.5}{4.8.12} \times \frac{z^6}{a^6} + \&c.\right);$$

unde fiet

$$s = \frac{1}{\sqrt{2}} \int dx \left(\frac{a}{z} + \frac{1}{4} \times \frac{z}{a} + \frac{1.3}{4.8} \times \frac{z^3}{a^3} + \frac{1.3.5}{4.8.12} \times \frac{z^5}{a^5} + \&c.\right)$$

$$s - y = \frac{1}{\sqrt{2}} \int dx \left(\frac{z}{a} + \frac{1}{4} \times \frac{z^3}{a^3} + \frac{1.3}{4.8} \times \frac{z^5}{a^5} + \frac{1.3.5}{4.8.12} \times \frac{z^7}{a^7} + \&c.\right).$$

24. Quia autem hæc integralia tantum pro caſu $x = c$ deſideramus; quo caſu fit $z = 0$, ea commode ope peripheriæ Circuli exprimi poterunt. Poſita enim ratione diametri ad peripheriam $= 1 : \pi$, erit $\int \frac{dx}{z} = \int \frac{dx}{\sqrt{(cc - xx)}} = \frac{\pi}{2}$; poſito poſt integrationem $x = c$. Pari modo autem ſequentia integralia ita determinabuntur, ut ſit

$$\int z\,dx = \frac{1}{2} \times \frac{\pi}{2}\,cc$$

$$\int z^3 dx = \frac{1.3}{2.4} \times \frac{\pi}{2}\,c^4$$

$$\int z^5 dx = \frac{1.3.5}{2.4.6} \times \frac{\pi}{2}\,c^6$$

$$\int z^7 dx = \frac{1.3.5.7}{2.4.6.6} \times \frac{\pi}{2}\,c^8$$

&c.

His ergo integralibus in ſubſidium vocatis, erit:

$$AC = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1.1}{2.2} \times \frac{cc}{2aa} + \frac{1.1.3.3}{2.2.4.4} \times \frac{c^4}{4a^4} + \&c.\right)$$

$$AC - AD = \frac{\pi a}{2\sqrt{2}}\left(\frac{cc}{2aa} + \frac{1.3}{2.4} \times \frac{c^4}{4a^4} + \frac{1.3.3.5}{2.4.4.6} \times \frac{c^6}{8a^6} + \&c.\right).$$

Ex his ergo reperiuntur AD & AC ut ſequitur:

$$AC = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1^2}{2^2} \times \frac{cc}{2aa} + \frac{1^2.3^2}{2^2.4^2} \times \frac{c^4}{4a^4} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} \times \frac{c^6}{8a^6} + \&c.\right)$$

$$AD = \frac{\pi a}{2\sqrt{2}}\left(1 - \frac{1^2}{2^2} \times \frac{3}{1} \times \frac{cc}{2aa} - \frac{1^2.3^2}{2^2.4^2} \times \frac{5}{3} \times \frac{c^4}{4a^4} - \frac{1^2.3^2.5^2}{2^2.4^2.6^2} \times \frac{7}{5} \times \frac{c^6}{8a^6} - \&c.\right).$$

Si itaque detur $AE = c$, & $AD = b$, ex his æquationibus & recta conſtans a & longitudo curvæ AC definietur. Viciſſim autem ex data longitudine curvæ AC, & recta a, per quam vis inflectens determinatur, reperiri poterunt rectæ AD & CD.

Species prima. 25. Quoniam igitur ſpeciem primam conſtituimus, ſi in æquatione generali $dy = \frac{(aa - cc + xx)\,dx}{\sqrt{(cc - xx)(2aa - cc + xx)}}$ fuerit $c = 0$, ſeu $\frac{a}{c} = \infty$, quo caſu linea reſultat repræſentans ſtatum laminæ Elaſticæ naturalem; ad eandem ſpeciem primam referamus quoque eos caſus, quibus c eſt quantitas quamminima, ita ut præ a pro evaneſcente haberi queat. Quia ergo x ipſam c ſuperare nequit; etiam x præ a evaneſcet, ideoque iſta prodibit æquatio $dy = \frac{a\,dx}{\sqrt{2(cc - xx)}}$, cujus integrale eſt $y = \frac{a}{\sqrt{2}}\,\text{A ſin.}\,\frac{x}{c}$, quæ eſt æquatio pro curva Trochoide in infinitum elongata. Fiet autem $AD = \frac{\pi a}{2\sqrt{2}}$, a qua ipſa curvæ longitudo infinite parum tantum diſcrepat, propterea quod angulus DAM eſt infinite parvus. Sit longitudo laminæ ACB $= 2f$, ejuſque elaſticitas abſoluta $= Ekk$; ob $f = \frac{\pi a}{2\sqrt{2}}$, erit vis ad hanc curvaturam infinite parvam laminæ inducendam requi-

requiſita finitæ magnitudinis & quidem $= \frac{Ekk}{ff} \times \frac{\pi\pi}{4}$. Scilicet ſi extremitates A & B colligentur filo AB, hoc filum contrahi debebit vi $= \frac{Ekk}{ff} \times \frac{\pi\pi}{4}$.

26. Secundam ſpeciem conſtituat caſus, quo $c > o$, attamen $c < a$; ſcilicet ſi c contineatur intra limites o & a. His enim caſibus angulus DAM recto erit minor; eſt namque anguli PAM ſinus, ſeu anguli DAM coſinus $= \frac{aa - cc}{aa}$. Hoc ergo caſu, forma lineæ curvæ talis fere erit qualem Figura 6, repræſentat. Quia igitur eſt $c < a$ erit $\frac{cc}{2aa} < \frac{1}{2}$; cum vero ſit $\frac{cc}{2aa} > o$, erit utique $AC = f > \frac{\pi a}{2\sqrt{2}}$ unde $aa < \frac{8ff}{\pi\pi}$; quare vis, qua extremitates laminæ A & B ope fili AB ad ſe invicem attrahuntur, major erit quam caſu præcedente, nempe $> \frac{Ekk}{ff} \times \frac{\pi\pi}{4}$. *Species ſecunda.*

27. In tertia ſpecie unicum complector caſum, quo $c = a$, quia hoc caſu axis AP curvam in puncto A tangit: hæcque ſpecies ſingulare nomen curvæ Elaſticæ rectangulæ obtinuit. Erit ergo $dy = \frac{xxdx}{\sqrt{(a^4 - x^4)}}$, & $ds = \frac{aadx}{\sqrt{(a^4 - x^4)}}$; hoc igitur caſu AD & AC ita ſe habebunt ut ſit: *Species tertia.*

$$AC = f = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1^2}{2^2} \times \frac{1}{2} + \frac{1^2.3^2}{2^2.4^2} \times \frac{1}{4} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} \times \frac{1}{8} + \&c.\right)$$

$$AD = b = \frac{\pi a}{2\sqrt{2}}\left(1 - \frac{1^2}{2^2} \times \frac{3}{1.2} - \frac{1^2.3^2}{2^2.4^2} \times \frac{5}{3.4} - \frac{1^2.3^2.5^2}{2^2.4^2.6^2} \times \frac{7}{5.8} - \&c.\right).$$

Quanquam autem hinc, neque b, neque f per a accurate aſſignari poteſt; tamen alibi inſignem relationem inter has quantitates locum habere demonſtravi. Scilicet oſtendi eſſe $4bf = \pi aa$, ſeu rectangulum ex AD & AC formatum erit æquale areæ Circuli cujus diameter eſt $=$ AE. Reperietur autem, calculum ſubducendo, proxime $f = \frac{5a}{6} \times \frac{\pi}{2}$, ita ut ſit $a = \frac{12f}{5\pi}$; hinc vis

Digitized by Google

qua laminæ extremitates A, B ad ſe invicem contrahi debent; erit $= \frac{Ekk}{ff} \times \frac{25}{72} \pi\pi$. Propius vero reperitur $f = \frac{\pi a}{2\sqrt{2}} \cdot 1,1803206$, hincque $b = \frac{\pi aa}{4f} = \frac{a}{\sqrt{2}} \times 1,1803206$; unde in numeris puris erit $\frac{f}{a} = 1,311006$. & $\frac{b}{a} = 0,834612$.

Species quarta

28. Si $c > a$, orietur ſpecies quarta, eouſque patens, quoad fiat $AD = b = 0$; qui alter limes ipſius c definietur per hanc æquationem:

$$1 = \frac{1^2}{2^2} \times \frac{3}{1} \times \frac{cc}{2aa} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \times \frac{5}{3} \times \frac{c^4}{4a^4} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \times \frac{7}{5} \times \frac{c^6}{8a^6} + \&c.$$

Fig. 7.

In hac ergo ſpecie cum ſit $c > a$; curva in A ſupra axem AE aſcendet, angulumque conſtituet PAM, cujus ſinus erit $= \frac{cc - aa}{aa}$; mox autem videbimus hunc angulum PAM minorem eſſe quam 40°, 41′; quoniam ſi hunc valorem acquirit, intervallum AD evaneſcit, quem caſum ad ſpeciem quintam refero. Hinc in ſpecie quarta continentur caſus quibus $\frac{cc}{aa}$ inter hos limites 1 & 1,651868 comprehenditur. Harum autem curvarum forma ex figura intelligitur; dummodo notetur, quo propius $\frac{cc}{aa}$ ad poſteriorem limitem 1,651868 acceſſerit; eo minus eſſe futurum intervallum AD, eoque propius laminæ terminos A & B ad ſe invicem adduci. Fieri ergo poteſt ut laminæ gibboſitates m & R, item M & r, ſe mutuo non ſolum tangant, ſed etiam interſecent, atque hujuſmodi interſectiones in infinitum multiplicabuntur, donec omnes diametri DC, dc coincidant, atque cum axe AE confundantur.

Species quinta. Fig. 8.

29. Hoc ſi evenerit, orietur ſpecies quinta, cujus natura hac exprimetur æquatione inter coordinatas $AP = x$ & $PM = y$; $dy = \frac{(cc - aa - xx)dx}{\sqrt{(cc - xx)(2aa - cc + xx)}}$, exiſtente hac inter a & c relatione, ut ſit intervallum $AD = b = 0$. Ponatur

$\frac{cc}{2aa}$

Tabula III. *Additamentum.*

Fig. 2.

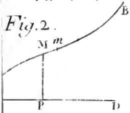


Fig. 3.

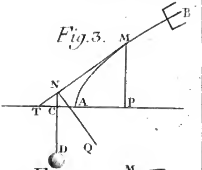


Fig. 5.

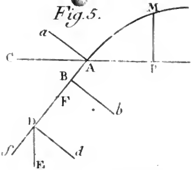


Fig. 6.

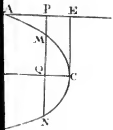
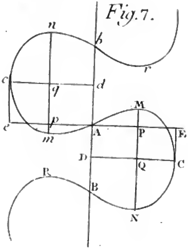


Fig. 7.



Digitized by Google

$\frac{cc}{2aa} = v$, atque v ex hac æquatione infinita definiri debet

$$1 = \frac{1.3}{2.2}v + \frac{1.1.3.5}{2.2.4.4}v^2 + \frac{1.1.3.3.5.7}{2.2.4.4.6.6}v^3 + \&c.$$

Quærantur primum per methodos cuique ſolitas, vel ſaltem tentando, limites inter quos verus valor ipſius v contineatur, atque hujuſmodi limites reperientur $v = 0,824$ & $v = 0,828$. Quod ſi jam uterque ſubſtituatur in æquatione ex erroribus binis oriundis, concludetur tandem fore $v = 0,825934 = \frac{cc}{2aa}$; unde fit $\frac{cc}{aa} = 1,651868$ & $\frac{cc - aa}{aa} = 0,651868$; quæ expreſſio cum ſit ſinus anguli PAM, ex Tabulis reperietur hic angulus $= 40°, 41'$; ideoque hujus duplum, ſeu angulus MAN, erit $= 81°, 82'$. Quare ſi laminæ elaſticæ extremitates eouſque ad ſe invicem adducantur, ut ſe contingant; tum curvam AMCNA formabunt, & ambæ extremitates in A angulum conſtituent $= 81°, 22'$.

30. Si ambæ extremitates laminæ A & B, poſtquam ad ſe invicem fuerint adductæ, aucta vi in plagas contrarias a ſe invicem diducantur; orietur curva hujus formæ AMCNB, quæ ſpeciem ſextam conſtituat. In curvis ergo ad hanc ſpeciem pertinentibus, erit $\frac{cc}{2aa} > 0,825934$; ita tamen ut ſit $\frac{cc}{2aa} < 1$. Quod ſi enim ſit $cc = 2aa$ orietur ſpecies ſeptima mox explicanda. Erit ergo in his curvis angulus PAM, quem curva in A cum axe conſtituit major quam $40°, 41'$, minor tamen recto: cum enim ejus ſinus ſit $= \frac{cc - aa}{aa}$, ob $cc < 2aa$, ſinus iſte neceſſario eſt minor ſinu toto; neque ergo angulus PAM rectus fieri poteſt, niſi ponatur $cc = 2aa$. *Species ſexta. Fig. 9.*

31. Sit jam $cc = 2aa$, quo caſu ſpecies ſeptima conſtituitur, atque natura curvæ exprimetur hac æquatione *Species ſeptima.*

$dy = \frac{(aa - xx)\,dx}{x\sqrt{(2aa - xx)}}$; ex qua colligitur, curvæ ramos A & B infinitum extendi ita, ut recta AB fiat curvæ aſymtota. *Fig. 10.* Fiet ergo uterque ramus AMC & BNC infinitus, id quod ex ſerie

Digitized by Google

ſerie ſupra pro arcu AC inventa intelligitur; erit enim $AC = \frac{\pi a}{2\sqrt{2}} \left(1 + \frac{1^2}{2^2} + \frac{1^2 . 3^2}{2^2 . 4^2} + \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} + \&c.\right)$, cujus ſeriei ſumma eſt infinita. Quod ſi igitur laminæ longitudo AC fuerit finita $= f$, neceſſe eſt ut ſit $a = 0$, hincque etiam $CD = c = 0$; lamina ergo, poſtquam in nodum fuerit incurvata, hoc caſu iterum in directum extendetur, ad quam extenſionem opus erit vi infinita. Sin autem lamina fuerit infinite longa, curvam formabit nodatam ad aſymtotam AB convergentem, exiſtente $CD = c$. Æquatio autem pro hac curva ope logarithmorum integrari poteſt, obtinebitur enim

$$y = \sqrt{(cc - xx)} - \frac{c}{2}\, l\, \frac{c + \sqrt{(cc - xx)}}{x},$$

ſumptis abſciſſis x in ipſa diametro DC; ita ut ſit $DQ = x$, & $QM = y$; evaneſcit enim applicata y, poſito $x = CD = c$. In nodo autem O applicata y pariter evaneſcit: ad quem locum inveniendum, ponatur $\frac{2\sqrt{(cc - xx)}}{c} = l\, \frac{c + \sqrt{(cc - xx)}}{x}$. Sit φ angulus cujus coſinus $= \frac{x}{c}$ & ſinus $= \frac{\sqrt{(cc - xx)}}{c}$, erit $2\,\text{ſin.}\, \varphi = l\, \text{tang.}\, (45^\circ + \frac{1}{2}\varphi)$, qui logarithmus ex hyperbolicorum genere ſumi debet; cujuſmodi Canon ſi deficiat, ſumatur ex Canone vulgari logarithmus tangentis anguli $45^\circ + \frac{1}{2}\varphi$, a cujus characteriſtica denarius auferatur, ſitque reſiduum $= \omega$; quo facto erit $2\,\text{ſin.}\, \varphi = \omega . 2,30258509$: ſumendis ergo iterum logarithmis vulgaribus, erit $l2 + l\,\text{ſin.}\, \varphi = l\omega + 0,3622156886$, ſeu $l\,\text{ſin.}\, \varphi = l\omega + 0,0611856930$. Hoc artificio tentando, mox vero proximus valor anguli φ elicietur; unde porro per regulam falſi verus valor anguli φ, ex eoque abſciſſa $x = DO$ definietur. Reperitur autem hoc modo angulus $\varphi = 73^\circ, 14', 12''$, unde prodit $\frac{x}{c} = 0,2884191$, & $\frac{\sqrt{(cc - xx)}}{c} = 0,9575042$; angulus vero QOM fit $= 2\varphi - 90 = 56^\circ, 28', 24''$, ideoque angulus MON $= 112^\circ, 56', 48''$. Cum igitur ſpecie quinta angulus nodi eſſet $81^\circ, 22'$, in

in ſpecie ſexta angulus nodi MON continebitur inter limites 81°, 22′ & 112°, 56′, 48″. In ſpecie quarta autem ſiquidem detur nodus, erit ejus angulus minor quam 81°, 22′.

32. Sit jam $cc > 2aa$, puta $cc = 2aa + gg$; erit æquatio pro curva, ob $aa = \frac{cc - gg}{2}$ *Species octava.*

$$dy = \frac{(xx - \frac{1}{2}cc - \frac{1}{2}gg)\,dx}{\sqrt{(cc - xx)(xx - gg)}},$$ qua æquatione ſpecies octava continetur, eritque, ſi recta dDd repræſentet directionem vis ſollicitantis, $x = DQ$ & $y = QM$. *Fig. 14.* Primum ergo patet applicatam y realem eſſe non poſſe, niſi ſit $x > g$, tum vero x non poteſt excedere rectam $DC = c$, unde ſumpta $DF = g$, tota curva continebitur inter rectas ipſi dd parallelas per puncta C & F ductas, quæ curvam ſimul tangent. Perinde autem eſt utra rectarum c & g ſit major, dummodo fuerint inæquales; æquatio enim non variatur ſi rectæ c & g inter ſe permutentur. Deinde vero hæc curva quoque habebit infinitas diametros inter ſe parallelas DC, dc, dc, & quæ per ſingula puncta G & H ducuntur rectæ pariter ad dDd normales; nuſquam autem per totam curvam dabitur punctum flexus contrarii, ideoque continua curvatura utrimque in infinitum progredirur, uti figura indicat; anguli autem qui in nodis conſtituuntur MON majores erunt quam 112°, 56′, 42″.

33. Cum in hac ſpecie non ſolum contineantur caſus quibus $gg < cc$, ſed etiam quibus $gg > cc$, unicus adhuc ſupereſt caſus quo $c = g$, quo quidem tota curva in ſpatium evaneſcens, ob $CF = 0$, redigitur. *Species nona.* Quod ſi autem utramque c & g ſtatuamus infinitam, ita tamen ut earum differentia fiat finita, curva finitum ſpatium occupabit. Ad eam ergo inveniendam, ponatur $g = c - 2h$, & $x = c - h - t$, atque ob $c = \infty$, quantitates h & t vero finitæ, erit $\frac{1}{2}cc + \frac{1}{2}gg = cc - 2ch$; & $xx - \frac{1}{2}cc - \frac{1}{2}gg = -2ct$; tum vero $cc - xx = 2c(h + t)$, & $xx - gg = 2c(h - t)$; ex quibus ſequens prodibit æquatio, $dy = \frac{t\,dt}{\sqrt{(hh - tt)}}$, pro Circulo. Lamina

elaſtica ergo hoc caſu in Circulum incurvatur, uti ſuprà jam annotavimus; Circulus ergo ſpeciem nonam atque ultimam conſtituet.

Fig. 12. 34. His enumeratis ſpeciebus, facile erit pro quovis caſu oblato aſſignare, ad quamnam ſpeciem curva formata pertineat. Sit lamina elaſtica in G muro infixa, termino vero A appendatur pondus P, quo lamina in figuram GA incurvetur. Ducatur tangens AT, atque ex angulo TAP totum judicium erit petendum. Si enim hic angulus fuerit acutus, referetur curva ad ſpeciem ſecundam; ſin ſit rectus ad tertiam, eritque elaſtica rectangula. Quod ſi angulus TAP fuerit obtuſus, minor tamen quam 130°, 41′, curva ad ſpeciem quartam pertinebit; ad quintam autem ſi angulus TAP ſit $=$ 130°, 41′; ſin autem angulus TAP major fuerit, curva ſub ſpecie ſexta continebitur. Ad ſeptimam vero pertineret, ſi iſte angulus fieret duobus rectis æqualis; quod autem nunquam fieri poteſt. Hæc igitur ſpecies cum duabus ſequentibus produci nequit laminæ immediate pondus appendendo.

Fig. 3. 35. Ut igitur pateat quomodo reliquæ ſpecies laminam incurvando produci queant, laminæ in B fixæ, non immediate, ſed virgæ rigidæ AC cum laminæ termino A firmiſſime connexæ in C appendatur pondus P, quod trahat in directione CD. Sit intervallum $AC = h$, elaſticitas laminæ abſoluta $= Ekk$, & anguli MAP quem lamina in A cum horizontali conſtituit, ſinus $= m$. His poſitis, ſi ponatur abſciſſa $AP = t$ & applicata $PM = y$, reperietur pro curva iſta æquatio

$$dy = \frac{dt(mEkk - Pht - \frac{1}{2}Ptt)}{\sqrt{(E^2k^4 - (mEkk - Pht - \frac{1}{2}Ptt)^2)}}.$$

Ponatur jam $CP = x = h + t$, quo æquatio ad formam qua in diviſione ſpecierum uſi ſumus, reducatur; erit

$$dy = \frac{dx(mEkk + \frac{1}{2}Phh - \frac{1}{2}Pxx)}{\sqrt{(E^2k^4 - (mEkk + \frac{1}{2}Phh - \frac{1}{2}Pxx)^2)}}$$

quæ comparata cum forma

$$dy = \frac{dx(aa - cc + xx)}{\sqrt{(cc - xx)(2aa - cc + xx)}}$$

ſeu

ſeu $dy = \frac{dx(aa - cc + xx)}{\sqrt{(a^4 - (aa - cc + xx)^2)}}$ dabit $\frac{1}{2} Paa = Ekk$, ſeu $aa = \frac{2Ekk}{P}$; & $\frac{1}{2} Pcc - \frac{1}{2} Paa = mEkk + \frac{1}{2} Phh$; ergo $cc = \frac{2(1+m)Ekk}{P} + hh$.

36. Curva ergo ad ſpeciem ſecundam pertinebit, ſi fuerit $\frac{2mEkk}{P} + hh < 0$, ſeu $P < \frac{-2mEkk}{hh}$: niſi ergo angulus PAM ſit negativus, vis P negativa eſſe atque virga in C ſurſum trahi debet. Ad ſpeciem tertiam curvatura pertinebit, ſi $P = \frac{-2mEkk}{hh}$. Quarta autem ſpecies prodibit ſi fuerit $2mEkk + Phh > 0$, ſimul vero $2mEkk + Phh < 2\alpha Ekk$, exiſtente $\alpha = 0,651868$. Sin autem ſit $P = \frac{2(\alpha - m)Ekk}{hh}$, tum curva ad ſpeciem quintam pertinebit. Quod ſi vero fuerit $Phh > 2(\alpha - m)Ekk$, ſimul vero $Phh < 2(1 - m)Ekk$, curva ad ſpeciem ſextam eſt referenda. Septimaque ſpecies proveniet, ſi $Phh = 2(1 - m)Ekk$. Octava autem ſpecies obtinebitur, ſi $Phh > 2(1 - m)Ekk$; quare ſi angulus PAM fuerit rectus, ob $1 - m = 0$, curva ſemper ad ſpeciem octavam pertinebit. Species denique nona orietur, ſi fuerit $h = \infty$; uti jam ſupra annotavi.

37. Quæ ante de ſpecie prima ſunt annotata inſervire poſſunt viribus columnarum dijudicandis. Sit enim AB columna ſuper baſi A verticaliter poſita, geſtans pondus P. Quod ſi jam columna ita ſit conſtituta ut prolabi nequeat; ab onere P, ſi fuerit nimis magnum, nil aliud erit metuendum, niſi columnæ incurvatio; hoc ergo caſu columna ſpectari poterit tanquam elaſticitate prædita. Sit igitur elaſticitas abſoluta columnæ $= Ekk$: ejuſque altitudo $AB = 2f = a$, atque ſupra §. 25 vidimus, vim requiſitam ad hanc columnam vel minimum inclinandam eſſe $= \frac{\pi\pi Ekk}{4ff} = \frac{\pi\pi}{aa} \cdot Ekk$. Niſi ergo onus geſtandum P majus

De vi Columnarum. *Fig. 13.*

Digitized by Google

ſit quam $E.\frac{\pi\pi kk}{aa}$, nulla prorſus incurvatio erit metuenda; contra vero ſi pondus P fuerit majus, columna incurvationi reſiſtere non poterit. Manente autem elaſticitate columnæ, atque adeo ejus craſſitie eadem; pondus P, quod ſine periculo geſtare valet, erit reciproce ut quadratum altitudinis columnæ: columnaque duplo altior quartam tantum oneris partem geſtare poterit. Hæc igitur præcipue in uſum vocari poſſunt circa columnas ligneas, quippe quæ incurvationi ſunt obnoxiæ.

Elaſticitatis abſolutæ determinatio per experimenta. Fig. 14

38. Quo autem vis atque incurvatio cujuſque laminæ elaſticæ a priori determinari queat; neceſſe eſt ut elaſticitas abſoluta, quam hactenus per Ekk expreſſimus, ſit cognita; id quod unico experimento commode præſtabitur. Infigatur lamina elaſtica uniformis FH, cujus elaſticitatem abſolutam inveſtigari oportet, altero termino F parieti firmo GK; ita ut ſitum teneat horizontalem FH; hic enim gravitatem naturalem negligere liceat. Alteri termino H appendatur pondus pro arbitrio ſumptum P, quo lamina in ſtatum AF incurvetur. Sit longitudo laminæ $AF = HF = f$, recta horizontalis $AG = g$, & verticalis $GF = h$; qui valores omnes per experimentum erunt cogniti. Comparetur jam hæc curva cum æquatione generali

$$dy = \frac{(cc - aa - xx)\,dx}{\surd(cc - xx)(2aa - cc + xx)}$$

in qua ſi fuerint a & c per f, g, h, definita; erit vis incurvans $P = \frac{2Ekk}{aa}$; ideoque elaſticitas abſoluta $Ekk = \frac{1}{2}Paa$.

39. Quia jam tangens in F eſt horizontalis, erit hic $\frac{dy}{dx} = 0$; ideoque $x = \surd cc - aa$. Hinc ergo erit $AG = g = \surd(cc - aa)$, & $aa = cc - gg$; ideoque

$$dy = \frac{(gg - xx)\,dx}{\surd(cc - xx)(cc - 2gg + xx)}$$

poſito autem hic $x = g$, fieri debebit $y = GF = h$, ſeu $s = AF = f$; eſt vero $ds = \frac{(cc - gg)\,dx}{\surd(cc - xx)(cc - 2gg + xx)}$.

Jam ſi pondus P ſumatur valde parvum, ut lamina pauliſ-

per

per tantum deprimatur; tum erit c quantitas valde magna; ideoque erit proxime $\frac{1}{\surd(cc - xx)(cc - 2gg + xx)} =$ $(c^4 - 2ccgg + 2ggxx - x^4)^{-\frac{1}{2}} = \frac{1}{cc} + \frac{gg}{c^4} - \frac{ggxx}{c^6}$ $+ \frac{x^4}{2c^6}$, ideoque integrando quoque proxime:

$$s = \frac{(cc-gg)x}{cc} + \frac{(cc-gg)ggx}{c^4} - \frac{(cc-gg)ggx^3}{3c^6} + \frac{(cc-gg)x^5}{10c^6},$$

$$\& \, y = \frac{ggx}{cc} + \frac{g^4x}{c^4} - \frac{g^4x^3}{3c^6} + \frac{ggx^5}{10c^6}$$

$$- \frac{x^3}{3cc} - \frac{ggx^3}{3c^4} + \frac{ggx^5}{5c^6} - \frac{x^7}{14c^6}.$$

Sit nunc $x = g$, fietque $f = g - \frac{37g^5}{30c^4}$ & $h = \frac{2g^3}{3cc} + \frac{2g^5}{3c^4}$.

Quod ſi ergo recta FG $= h$ in uſum vocetur, erit $cc = \frac{2g^3}{3h}$; & $aa = \frac{g(2gg - 3gh)}{3h}$: unde elicitur elaſticitas abſoluta $Ekk = \frac{Pgg(2g - 3h)}{6h}$; qui valor a vero vix ſenſibiliter diſcrepabit, dummodo laminæ curvatura non nimis magna inducatur.

40. Hæc autem elaſticitas abſoluta Ekk primum pendet ab natura materiæ, ex qua lamina eſt fabrefacta; unde alia materia magis, alia minus elatere prædita dici ſolet. Secundo quoque ita pendet a laminæ latitudine, ut expreſſio Ekk ubique latitudini laminæ debeat eſſe proportionalis, ſi cetera ſint paria. Tertio verum craſſities laminæ plurimum confert ad valorem ipſius Ekk determinandum, quæ ita comparata eſſe videtur, ut, ceteris paribus, Ekk ſit ut craſſitiei quadratum. Conjunctim ergo tenebit expreſſio Ekk rationem compoſitam ex ratione elateris materiæ, latitudinis laminæ ſimplici, ac duplicata craſſitiei laminæ. Hinc per experimenta quibus latitudinem & craſſitiem

 metiri

metiri licet, omnium materiarum elaſticitates inter ſe comparari ac determinari poterunt.

De curvatura laminæ elaſticæ inæquabilis *Fig. 2.*

41. Quemadmodum igitur hactenus laminæ, cujus curvaturam determinavi, elaſticitatem abſolutam Ekk per totam longitudinem conſtantem poſui; ita ſolutio eadem methodo poterit abſolvi, ſi quantitas Ekk utcunque ponatur variabilis. Scilicet ſi elaſticitas abſoluta fuerit ut functio quæcunque portionis laminæ AM, quæ functio ſit $= S$, poſito arcu AM $= s$; atque exiſtente radio oſculi in M $= R$; curva AM, quam lamina induit, ita erit comparata, ut in ea, inter omnes alias ejuſdem longitudinis, ſit $\int \frac{Sds}{RR}$ minimum. Solvetur ergo iſte caſus per formulam ſecundam generalem. Sit $dy = pdx$; $dp = qdx$; at $dS = Tds$, atque, inter omnes curvas in quibus eſt $\int dx \surd(1+pp)$ ejuſdem magnitudinis, ea determinari debebit, in qua ſit $\int \frac{Sqqdx}{(1+pp)^{5:2}}$ minimum. Prior formula $\int dx\surd(1+pp)$ dat pro formula differentiali $\frac{1}{dx}\, d.\, \frac{p}{\surd(1+pp)}$. Altera vero $\int \frac{Sqqdx}{(1+pp)^{5:2}}$ cum $\int Zdx$ comparata dabit $Z = \frac{Sqq}{(1+pp)^{5:2}}$. Cum igitur poſitum ſit $dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq$, $\Pi = \int [Z]\, dx$, & $d[Z] = [M]dx + [N]dy + [P]dp$; erit $Ld\Pi = \frac{qqTds}{(1+pp)^{5:2}}$; unde $L = \frac{qqT}{(1+pp)^{5:2}}$; $d\Pi = ds = dx\surd(1+pp)$; ideoque $[Z] = \surd(1+pp)$, $[M] = 0$, $[N] = 0$, & $[P] = \frac{p}{\surd(1+pp)}$. Deinde vero eſt $M = 0$, $N = 0$, $P = -\frac{5Sqqpdp}{(1+pp)^{7:2}}$, & $Q = \frac{2Sq}{(1+pp)^{5:2}}$; ita ut ſit $dZ = \frac{qqdS}{(1+pp)^{5:2}} + Pdp + Qdq$.

42. Jam ſumatur integrale $\int Ldx = \int \frac{qqTdx}{(1+pp)^{5:2}} =$

$\int$

Digitized by Google

$\int \frac{qqdS}{(1+pp)^3}$, ſitque H ejus valor, ſi ponatur $x = a$, cujus quidem conſtantis a conſideratio mox ex calculo rurſus evaneſcet. Erit ergo $V = H - \int \frac{qqdS}{(1+pp)^3}$: Unde valor differentialis fiet $= -\frac{dP}{dx} - \frac{1}{dx} d. [P] V + \frac{ddQ}{dx^2}$. Quamobrem ex his duobus valoribus differentialibus naſcetur hæc æquatio pro curva quæſita

$$\frac{\alpha}{dx} d. \frac{p}{\sqrt{(1+pp)}} = + \frac{dP}{dx} + \frac{1}{dx} d. [P] V - \frac{ddQ}{dx^2},$$

quæ integrata dat

$$\frac{\alpha p}{\sqrt{(1+pp)}} + 6 = P + [P] V - \frac{dQ}{dx}, \text{ ſive}$$

$$\frac{\alpha p}{\sqrt{(1+pp)}} + 6 = \frac{Hp}{\sqrt{(1+pp)}} - \frac{p}{\sqrt{(1+pp)}} \int \frac{qqdS}{(1+pp)^3} + P - \frac{dQ}{dx},$$

ubi conſtans H alias determinata in conſtante arbitraria α comprehendi poteſt, quo ipſo conſtans a ex calculo egreditur. Idcirco ergo prodibit hæc æquatio

$$\frac{\alpha p}{\sqrt{(1+pp)}} + 6 = P - \frac{dQ}{dx} - \frac{p}{\sqrt{(1+pp)}} \int \frac{qqdS}{(1+pp)^3}.$$

43. Multiplicetur hæc æquatio per $dp = qdx$, atque prodibit:

$$\frac{\alpha p dp}{\sqrt{(1+pp)}} + 6dp = Pdp - Qdq - \frac{pdp}{\sqrt{(1+pp)}} \int \frac{qqdS}{(1+pp)^3}.$$

Cum autem ſit $dZ = \frac{qqdS}{(1+pp)^{5:2}} + Pdp + Qdq$, erit $Pdp = dZ - Qdq - \frac{qqdS}{(1+pp)^{5:2}}$; quo valore ſubſtituto; emerget æquatio integrabilis hæc:

$$\frac{\alpha p dp}{\sqrt{(1+pp)}} + 6dp = dZ - qdQ - Qdq - \frac{qqdS}{(1+pp)^{5:2}} - \frac{pdp}{\sqrt{(1+pp)}} \int \frac{qqdS}{(1+pp)^3},$$

cujus integralis eſt:

$$\alpha \sqrt{(1+pp)} + 6p + \gamma = Z - Qq - \sqrt{(1+pp)} \int \frac{qqdS}{(1+pp)^3}, \text{ ſeu}$$

$$\alpha\sqrt{(1+pp)}+\beta p+\gamma=\frac{-Sqq}{(1+pp)^{5:2}}-\sqrt{(1+pp)}\int\frac{qqdS}{(1+pp)^{3}}.$$

Quo ſignum integrale tollamus, diviſa æquatione per $\sqrt{(1+pp)}$, ea denuo differentietur;

$$\frac{\beta dp}{(1+pp)^{3:2}}-\frac{\gamma p dp}{(1+pp)^{3:2}}+\frac{2qqdS}{(1+pp)^{3}}+\frac{2Sqdq}{(1+pp)^{3}}-\frac{6Spqqdp}{(1+pp)^{4}}=0$$

quæ per $\frac{(1+pp)^{3:2}}{2q}$ multiplicatâ, præbet:

$$\frac{\beta dp}{2q}-\frac{\gamma p dp}{2q}+\frac{qdS+Sdq}{(1+pp)^{3:2}}-\frac{3Spqdp}{(1+pp)^{5:2}}=0;$$

cujus, ob $dp=qdx$ & $dy=pdx$, integrale erit

$$\alpha+\tfrac{1}{2}\beta x-\tfrac{1}{2}\gamma y+\frac{Sq}{(1+pp)^{3:2}}=0.$$

At eſt $\frac{-(1+pp)^{3:2}}{q}$ = radio oſculi R; unde conſtantes β & γ duplicando, orietur hæc æquatio $\frac{S}{R}=\alpha+\beta x-\gamma y$; quæ æquatio apprime congruit cum ea, quam altera Methodus directa ſuppeditat. Exprimet enim $\alpha+\beta x-\gamma y$ momentum potentiæ incurvantis, recta quacunque pro axe aſſumpta, cui momento utique æqualis eſſe debet elaſticitas abſoluta S per radium oſculi R diviſa. Sic igitur non ſolum Celeb. BERNOULLII obſervata proprietas Elaſticæ pleniſſime eſt evicta; ſed etiam formularum mearum difficiliorum uſus ſummus in hoc Exemplo eſt declaratus.

44. Si ergo curva fuerit data, quam lamina inæquabiliter Fig. 3. elaſtica a potentia CD $=P$, ſollicitata format; hinc elaſticitas abſoluta laminæ in quovis loco poterit cognoſci. Sumpta enim recta CP, quæ ad directionem vis ſollicitantis eſt normalis, pro axe, ac poſita CP $=x$, PM $=y$, arcu curvæ AN $=s$, & radio oſculi in M $=R$, ob momentum potentiæ P ad punctum M relatum $=Px$; erit $\frac{S}{R}=Px$; ideoque elaſticitas abſoluta in puncto M, quæ eſt S, $=PRx$. Hinc cum,

cum, data curva, in ſingulis punctis detur radius oſculi R, elaſticitas abſoluta in quovis loco innoteſcit. Quod ſi ergo materia laminæ, una cum craſſitie, ubique fuerit eadem; latitudo autem ſit variabilis: quia elaſticitas abſoluta latitudini eſt proportionalis, ex curva formata latitudo laminæ in ſingulis locis colligitur.

45. Sit ex lamina elaſtica exciſſa lingula triangularis fAf, Fig 15.
ubique ejuſdem craſſitiei. Quoniam ergo latitudo mm, in quovis loco M, eſt longitudini AM proportionalis; poſita $AM = s$, erit elaſticitas abſoluta in M ut s. Sit ea $= Eks$; atque laminæ termino ff muro horizontaliter infixo, appendatur cuſpidi A pondus P; quo laminæ recta media AF in curvam FmA Fig 14.
incurvetur, cujus natura quæritur. Poſitis autem in axe horizontali abſciſſa $Ap = x$, applicata $pm = y$, & arcu $Am = s$; erit $Px = \frac{Eks}{R}$, denotante R radium oſculi in m. Multiplicetur hæc æquatio per dx, & ob $R = \frac{-ds^3}{dxddy}$, poſito dx conſtante, erit $Pxdx = \frac{-Eksdx^2ddy}{ds^3}$, ſeu $\frac{Pxdx}{Ek} + \frac{sdx^2ddy}{ds^3} = 0$. At, cum ſit $d.\frac{sdy}{ds} = \frac{sddy}{ds} - \frac{sdydds}{ds^2} + dy = \frac{sdx^2ddy}{ds^3} + dy$, ob $dds = \frac{dyddy}{ds}$, erit $\int\frac{sdx^2ddy}{ds^3} = \frac{sdy}{ds} - y$: unde integrando habebitur $\frac{Pxx}{2Ek} + a = \frac{-sdy}{ds} + y$.

46. Sit $dy = pdx$, erit $ds = dx\sqrt{(1+pp)}$, & poſito $\frac{2Ek}{P} = c$, fiet $a + \frac{xx}{c} = y - \frac{sp}{\sqrt{(1+pp)}}$; ideoque erit $\frac{a\sqrt{(1+pp)}}{p} + \frac{xx\sqrt{(1+pp)}}{cp} = \frac{y\sqrt{(1+pp)}}{p} - s$; quæ differentiata dat $\frac{-adp}{pp\sqrt{(1+pp)}} + \frac{2xdx\sqrt{(1+pp)}}{cp} - \frac{xxdp}{cpp\sqrt{(1+pp)}} = \frac{dy\sqrt{(1+pp)}}{p} - \frac{ydp}{pp\sqrt{(1+pp)}} - dx\sqrt{(1+pp)} = \frac{-ydp}{pp\sqrt{(1+pp)}}$. Hinc oritur $a - y = \frac{2pxdx(1+pp)}{cdp} - \frac{xx}{c}$. Ponatur dp conſtans,

ac differentiando erit $-pdx = \frac{2px\,ddx(1+pp)}{cdp} + \frac{2pdx^2(1+pp)}{cdp} + \frac{2xdx(1+3pp)}{c} - \frac{2xdx}{c}$, ſeu

$$0 = cdxdp + 2xddx(1+pp) + 2dx^2(1+pp) + 6pxdx;$$

cujus æquationis autem reſolutio ulterior non conſtat. Simpliciſſima autem pro curva eſt æquatio hæc $\frac{yds - sdy}{ds} = \frac{Pxx}{2Ek}$; quia enim poſito $x = 0$, & y & s evaneſcere debent, conſtans a debet eſſe $= 0$.

De Incurvatione laminarum elaſticarum naturaliter non rectarum.

47. Hoc igitur modo curvatura laminæ, ſive æqualiter ſive inæqualiter elaſticæ, determinatur, ſi ab una potentia ſollicitetur; atque, quod præcipue eſt notandum, ſi lamina naturaliter fuerit in directum extenſa. Quod ſi enim lamina in ſtatu naturali jam fuerit curva; tum utique a vi ſollicitante aliam curvaturam induet; ad quam inveniendam, præter ſollicitationem atque elaſticitatem, ſimul figuram ejus naturalem noſſe oportet. Sit igitur lamina elaſtica naturaliter curva B m a, cujus quidem elaſticitas ſit ubique eadem, $= Ekk$; quæ a vi ſollicitante P in figuram BMA incurvetur. Per A ducatur recta CAP ad directionem vis ſollicitantis normalis, quæ habeatur pro axe; ſitque intervallum AC $= c$, abſciſſa AP $= x$, applicata PM $= s$; erit momentum vis ſollicitantis pro puncto M $= P(c+x)$. *Fig. 16.*

48. Sit porro radius oſculi curvæ quæſitæ in M $= R$: ſumatur in ſtatu naturali arcus a m $=$ AM $= s$, ſitque in puncto m radius oſculi $= r$; qui ob curvam a m B cognitam dabitur per arcum s. In M ergo, quia curvatura major eſt, radius oſculi R minor eſt quam r, atque exceſſus anguli elementaris in M ſupra angulum in ſtatu naturali erit $= \frac{ds}{R} - \frac{ds}{r}$, qui exceſſus erit effectus a potentia ſollicitante productus. Quamobrem erit $P(c+x) = Ekk\left(\frac{1}{R} - \frac{1}{r}\right)$; quæ cum r per s detur, erit æquatio pro curva quæſita; quæ autem ſic in genere ſpectata ulterius reduci non poteſt.

49. Po-

49. Ponamus ergo laminam in ſtatu naturali amB habere figuram circularem; erit r radius ejus circuli qui ſit $= a$, unde fit $P(c+x) = Ekk(\frac{1}{R} - \frac{1}{a})$. Multiplicetur hæc æquatio per dx, & integretur; orietur $\frac{P}{Ekk}(\frac{1}{2}xx + cx + f) = -\frac{dy}{ds} - \frac{x}{a}$: quæ æquatio, ſi loco c ſcribatur $c + \frac{Ekk}{Pa}$; abibit in $\frac{P}{Ekk}.(\frac{1}{2}xx + cx + f) = \frac{-dy}{ds}$: quæ eſt eadem æquatio quam ſupra pro lamina naturaliter recta invenimus. Lamina ergo naturaliter circularis in eaſdem curvas incurvatur, quæ laminæ naturaliter rectæ inducuntur: tantum ſcilicet locus applicationis potentiæ, ſeu intervallum AC $= c$, pro utroque caſu ſecundum datam legem variari debebit. Eædem ergo novem ſpecies curvarum prodibunt pro figuris, quas lamina naturaliter circularis inducere poteſt, quas ſupra numeravimus. Lamina enim circularis, ſi intervallum AC capiatur infinitum, primum in lineam rectam extendi poteſt; tum quæcunque potentia inſuper applicata eundem præſtabit effectum, ac ſi ſola laminæ elaſticæ naturaliter rectæ applicaretur.

50. Ponamus autem, quæcunque ſit laminæ figura naturalis, punctum C infinite diſtare; ita ut momentum vis ſollicitantis ubique ſit idem, quod per Ekk diviſum ponatur $= \frac{1}{b}$; eritque $\frac{1}{b} = \frac{1}{R} - \frac{1}{r}$ & $\frac{1}{R} = \frac{1}{b} + \frac{1}{r}$. Hinc fiet $\int\frac{ds}{R} = \frac{s}{b} + \int\frac{ds}{r} =$ amplitudini arcus AM; ſicuti $\int\frac{ds}{r}$ exprimit amplitudinem arcus am; quemadmodum quidem Celeb. *Joh.* BERNOULLI hoc *amplitudinis* nomine in eximio Tractatu *De motu reptorio* uti eſt ſolitus. Sit igitur $\frac{s}{b} + \int\frac{ds}{r}$ arcus in circulo, cujus radius $= 1$ ſumptus, qui ob r per s datum, quoque in s erit cognitus. Hinc autem reperientur coordinatæ orthogonales x & y, ita ut ſit

 $x =$

$$x = \int ds\ \text{ſin.} \left(\frac{s}{b} + \int \frac{ds}{r}\right),\ \&\ y = \int ds\ \text{coſ.} \left(\frac{s}{b} + \int \frac{ds}{r}\right);$$

unde curva quæſita per quadraturas conſtrui poterit.

Fig. 17. 51. Hinc determinari poteſt figura a m B, quam lamina in ſitu naturali habere debet, ut a potentia P in directione AP ſollicitante in lineam rectam AMB explicetur. Sumpta enim longitudine AM $= s$; erit momentum potentiæ ſollicitantis pro puncto M $= Ps$; radius oſculi autem in M, per hypotheſin, erit infinitus ſeu $\frac{1}{R} = 0$. Sumpto jam in ſtatu naturali arcu a m $= s$, poſitoque radio oſculi in m $= r$; quia hæc curva convexitate ſua rectam AB ſpectat, in calculo præcedente poni debet r negativum. Hinc erit $Ps = \frac{Ekk}{r}$, ſeu $rs = aa$; quæ eſt æquatio naturam curvæ a m B complectens.

52. Cum igitur ſit $\frac{1}{r} = \frac{s}{aa}$; erit $\int \frac{ds}{r} = \frac{ss}{2aa}$; ſeu erit amplitudo arcus a m ut quadratum ipſius arcus. Hinc coordinatæ orthogonales x & y pro hac curva a m B ita definientur, ut ſit $x = \int ds$ ſin. $\frac{ss}{2aa}$, & $y = \int ds$ coſ. $\frac{ss}{2aa}$: Scilicet in circulo, cujus radius $= 1$, abſcindi debet arcus $= \frac{ss}{2aa}$, cujus ſinus & coſinus ad coordinatas determinandas aſſumi debent. Ex eo autem quod radius oſculi continuo decreſcit, quo major capiatur arcus a m $= s$, manifeſtum eſt curvam in infinitum non protendi, etiamſi arcus s capiatur infinitus. Curva ergo erit ex ſpiralium genere, ita ut infinitis peractis ſpiris in certo quodam puncto tanquam centro convolvatur, quod punctum ex hac conſtructione invenire difficillimum videtur. Non exiguum ergo analyſis incrementum capere exiſtimanda erit, ſi quis methodum inveniret, cujus ope, ſaltem vero proxime, valor horum integralium $\int ds$ ſin. $\frac{ss}{2aa}$ & $\int ds$ coſ. $\frac{ss}{2aa}$ aſſignari poſſet, caſu quo s ponitur infinitum; quod Problema non indignum videtur, in quo Geometræ vires ſuas exerceant.

53. Sit

Digitized by Google

53. Sit $2aa = bb$, & cum ſit

$$\text{fin.}\ \frac{ss}{bb} = \frac{s^2}{b^2} - \frac{s^6}{1.2.3b^6} + \frac{s^{10}}{1.2.3.4.5b^{10}} - \frac{s^{14}}{1.2\ldots7b^{14}} + \&c.$$

$$\text{coſ.}\ \frac{ss}{bb} = 1 - \frac{s^4}{1.2b^4} + \frac{s^8}{1.2.3.4b^8} - \frac{s^{12}}{1.2.3.4.5.6b^{12}} + \&c.$$

coordinatæ x & y curvæ quæſitæ commode per ſeries infinitas exprimi poterunt: erit enim

$$x = \frac{s^3}{1.3b^2} - \frac{s^7}{1.2.3.7b^6} + \frac{s^{11}}{1.2.3.4.5.11b^{10}} - \frac{s^{15}}{1.2\ldots7.15b^{14}} + \&c.$$

$$y = s - \frac{s^5}{1.2.5b^4} + \frac{s^9}{1.2.3.4.9b^8} - \frac{s^{13}}{1.2.3\ldots6.13b^{12}} + \&c.$$

ex quibus ſeriebus vehementer convergentibus, niſi arcus s aſſumatur valde magnus, valores coordinatarum x & y vero proxime ſatis expedite determinari poſſunt. Verum cujuſmodi valores x & y acquirant, ſi ponatur arcus s infinite magnus, ex his ſeriebus nullo modo concludi poteſt.

54. Quoniam igitur poſitio infiniti loco s facienda maximam parit difficultatem; huic quidem incommodo ſequenti modo remedium afferri poteſt. Ponatur $\frac{ss}{bb} = v$, ut ſit $s = b\sqrt{v}$, erit $ds = \frac{b\,dv}{2\sqrt{v}}$ fietque, $x = \frac{b}{2}\int\frac{dv}{\sqrt{v}}$ ſin. v, & $y = \frac{b}{2}\int\frac{dv}{\sqrt{v}}$ coſ. v. Nunc autem dico valores debitos pro x & y, ſi ponatur $s = \infty$, inventum iri ex his formulis integralibus,

$$x = \frac{b}{2}\int dv\left(\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{(\pi+v)}} + \frac{1}{\sqrt{(2\pi+v)}} - \frac{1}{\sqrt{(3\pi+v)}} + \&c.\right)\text{ſin.}\ v$$

$$y = \frac{b}{2}\int dv\left(\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{(\pi+v)}} + \frac{1}{\sqrt{(2\pi+v)}} - \frac{1}{\sqrt{(3\pi+v)}} + \&c.\right)\text{coſ.}\ v,$$

ſi poſt integrationem ponatur $v = \pi$, denotante π angulum duobus rectis æqualem. Hoc ergo modo poſitio infiniti quidem evitatur, contra vero ſeries infinita

$\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{(\pi+v)}} + \frac{1}{\sqrt{(2\pi+v)}} -$ &c. in calculum introducitur, cujus ſumma cum adhuc lateat, reſolutio hujus nodi maximæ adhuc difficultati eſt obnoxia.

Digitized by Google

De Incurvatione laminæ elaſticæ in ſingulis punctis a viribus quibuscunque ſollicitatæ.

55. Tradita jam methodo inveſtigandi curvaturam cujuſque laminæ elaſticæ, ſi ea ab una vi in dato loco applicata ſollicitetur; conveniet quoque curvaturam a pluribus, imo infinitis, potentiis laminæ elaſticæ inductam indagare. Quoniam vero nondum conſtat, cujuſmodi expreſſio his caſibus futura ſit vel maxima vel minima; methodo utar tantum directa, quo ex ipſa ſolutione fortaſſe proprietas ea, quæ eſt maxima vel minima erui queat. Sit igitur lamina elaſtica, naturaliter recta, in

Fig. 18. ſtatum A m M redacta, primum a viribus finitis P & Q ſecundum directiones CE & CF interſe normales ſollicitantibus, tum vero a viribus infinite parvis ſingulis laminæ elementis $m\mu$ applicatis, & ſecundum directiones mp & mq illis CF & CF parallelas trahentibus; quibus poſitis requiritur natura curvæ AmM laminæ inductæ.

56. Sumatur recta FCA producta pro axe, ponatur AC $= c$, & vocetur abſciſſa AP $= x$, applicata PM $= y$, arcus curvæ AM $= s$, & radius oſculi in M $= R$. Sit elaſticitas laminæ abſoluta conſtans $= Ekk$; atque ſumma momentorum ex omnibus viribus ſollicitantibus reſpectu puncti M ortorum æqualis eſſe debet $\frac{Ekk}{R}$. Primum quidem a vi finita P in directione CE trahente oritur momentum $= P(c+x)$, in eam plagam agens qua vis elaſtica æquilibratur. Momentum autem ex altera vi Q ortum, nempe Qy in contrariam plagam tendit, ex quo ex viribus finitis P & Q conjunctim oritur momentum $P(c+x) - Qy$. Jam conſideretur quodvis elementum laminæ intermedium $m\mu$, cujus reſpondens abſciſſa Ap ponatur $= \zeta$, & applicata pm $= \eta$, ſit autem vis elementum $m\mu$ in directione mp urgens $= dp$, & vis urgens in directione mq $= dq$; erit momentum ex his viribus pro puncto M ortum $= (x-\zeta)dp - (y-\eta)dq$.

57. Ad ſummam ergo omnium horum momentorum inveniendam, punctum M, ac proinde x & y, tantiſper pro conſtantibus haberi debent, dum ſolæ coordinatæ ζ & η cum viribus dp & dq tanquam variabiles ſpectantur. Erit ergo ſumma momen-

momentorum a viribus arcum Am ſollicitantibus ortorum $= xp - \int \zeta\, dp - yp + \int \eta\, dq$; ubi p exprimit ſummam omnium virium arcum AM in directionibus applicatis pm parallelis ſollicitantium, & q ſummam omnium virium arcum Am in directionibus axi Ap parallelis ſollicitantium. At eſt $\int \zeta\, dp = \zeta p - \int p\, d\zeta$ & $\int \eta\, dq = \eta q - \int q\, d\eta$; unde fit ſumma momentorum ex viribus arcui Am applicatis ortorum $= (x - \zeta)p + \int p\, d\zeta - (y - \eta)q - \int q\, d\eta$. Promoveatur jam punctum m in M uſque, fietque $\zeta = x$, $\eta = y$, & $d\zeta = dx$ atque $d\eta = dy$; unde ſumma omnium momentorum per totum arcum AM ſumptorum erit $= \int p\, dx - \int q\, dy$. Quocirca obtinebitur pro curva quæſita hæc æquatio $\frac{Ekk}{R} = P(c + x) - Qy + \int p\, dx - \int q\, dy$, ubi ergo p exprimit ſummam omnium virium verticalium ſeu in directionibus applicatarum MP agentium, & q ſummam omnium virium horizontalium ſeu in directionibus MQ axi AP parallelis agentium per totum arcum AM.

58. Si formula $p\, dx$ & $q\, dy$ integrationem non admittant, tum æquatio inventa per differentiationem ab his formulis integralibus liberari debebit, unde habebitur iſta æquatio:

$$\frac{-Ekk\, dR}{RR} = P\, dx - Q\, dy + p\, dx - q\, dy.$$

Sin autem nec p nec q per expreſſiones finitas exhiberi poſſint, quippe quæ jam exprimunt ſummas infinitarum virium infinite parvarum, tum per ulteriorem differentiationem valores finiti p & q exterminari debebunt, ut tantum inſint dp & dq cum differentio-differentialibus ddp & ddq. Orietur autem, poſt primam differentiationem, $-Ekk\, d.\frac{dR}{RR\, dx} = dp - (Q + q) \times d.\frac{dy}{dx} - \frac{dy}{dx}\, dq$. Sit $\frac{dy}{dx} = \omega$, eritque denuo æquatione differentiata:

$$-Ekk\, d.\frac{d.\frac{dR}{RR\, dx}}{d\omega} = d.\frac{dp}{d\omega} - 2\, dq - \omega\, d.\frac{dq}{d\omega}$$

quæ æquatio ad differentialia quarti ordinis aſcendit.

59. Sint

Digitized by Google

59. Sint laminæ, loco potentiarum verticalium & horizontalium p & q, in ſingulis punctis M applicatæ duæ potentiæ, altera normalis $MN = dv$ & altera tangentialis $MT = dt$. Hinc erit $dp = \frac{dxdv}{ds} + \frac{dydt}{ds}$ & $dq = \frac{dxdt}{ds} - \frac{dydv}{ds}$, &, ob $dy = \omega dx$ & $ds = dx\sqrt{(1+\omega\omega)}$, habebitur $dp = \frac{dv}{\sqrt{(1+\omega\omega)}} + \frac{\omega dt}{\sqrt{(1+\omega\omega)}}$, & $dq = \frac{dt}{\sqrt{(1+\omega\omega)}} - \frac{\omega dt}{\sqrt{(1+\omega\omega)}}$: quibus in præced. §. æquatione ultima ſubſtitutis, proveniet ſequens æquatio,

$$-Ekk\,d.\frac{d.\frac{dR}{RRdx}}{dv} = \frac{-dt}{\sqrt{(1+\omega\omega)}} + \frac{2\omega dv}{\sqrt{(1+\omega\omega)}} + \sqrt{(1+\omega\omega)}\,d.\frac{dv}{d\omega},$$

quæ multiplicata per $\sqrt{(1+\omega\omega)}$ fit integrabilis: poſito enim, brevitatis gratia, $z = \frac{dR}{RRdx}$, reperietur integrale,

$$A - t + \frac{dv(1+\omega\omega)}{d\omega} = -Ekk\left(\frac{dz\sqrt{(1+\omega\omega)}}{d\omega} - \frac{\omega z}{\sqrt{(1+\omega\omega)}} + \frac{1}{2RR}\right)$$

$$= -Ekk\left(\frac{1+\omega\omega}{d\omega}\,d.\frac{dR}{RRdx\sqrt{(1+\omega\omega)}} + \frac{1}{2RR}\right).$$

Cum vero ſit $R = \frac{-(1+\omega\omega)^{3:2}dx}{d\omega}$, erit $d\omega = \frac{-(1+\omega\omega)^{3:2}dx}{R}$; quo loco $d\omega$ valore ſubſtituto, habebitur:

$$A - t - \frac{Rdv}{ds} = -Ekk\left(\frac{1}{2RR} - \frac{R}{ds}\,d.\frac{dR}{RRds}\right),$$

ob $dx\sqrt{(1+\omega\omega)} = ds$. Quocirca æquatione ordinata, pro curva quæſita orietur hæc æquatio

$$t + \frac{Rdv}{ds} - A = Ekk\left(\frac{1}{2RR} - \frac{R}{ds}\,d.\frac{dR}{RRds}\right).$$

60. Primum quidem manifeſtum eſt, ſi vis elaſtica Ekk evaneſcat, laminam tranſmutari in filum perfecte flexile; atque hinc in his æquationibus continentur omnes curvæ, quas filum perfecte flexile a viribus quibuſcunque ſollicitatum formare poteſt. Sic ſi filum a propria gravitate tantum deorſum ſollicitatur, erit

$q = 0$,

$q = 0$, & p exprimet pondus funis AM, eritque ergo $\frac{p\,dx}{dy} = Q =$ conſtanti, facto $P = 0$, quæ eſt æquatio generalis pro omnis generis Catenariis. Sin autem filum perfecte flexile, in ſingulis punctis, a viribus quarum directiones ſunt normales ad ipſam curvam ſollicitetur, ita ut in puncto M filum ſollicitetur ſecundum directionem MN, vi $= dv$; ob $t = 0$, erit $\frac{R\,dv}{ds} = A =$ conſtanti, quæ eſt proprietas generalis curvarum Velariarum, Linteariarum, omniumque in quibus hujuſmodi ſollicitationes locum habent.

61. Ad laminas elaſticas autem revertor, de quibus mox iſta quæſtio præ ceteris notatu digna ſe offert, cujuſmodi figuram accipiat lamina elaſtica proprio pondere incurvata. Sit AmM hæc curva quæ quæritur, & quia ſolæ vires verticales a gravitate ortæ urgent, fiet $P = 0$, $Q = 0$, $q = 0$, & p exprimet pondus laminæ AM. Quare ſi F ſit pondus laminæ longitudinis a; quia lamina uniformis aſſumitur, erit $p = \frac{Fs}{a}$; unde curvæ natura hac exprimetur æquatione $\frac{-Ekk\,dR}{RR} = \frac{Fs\,dx}{a}$. Sit amplitudo curvæ $\int \frac{ds}{R} = u$, erit $R = \frac{ds}{du}$, & $dx = ds$ ſin. u; unde, ſumpto elemento ds conſtante, reperietur æquatio $s\,ds$ ſin. $u + \frac{Eakk}{F} \cdot \frac{ddu}{ds} = 0$, quæ autem, quantum primo intuitu patet, ulterius reduci nequit. *De curvatura laminæ elaſticæ a proprio pondere orta.*

62. In primis autem notari meretur curva, quam fluidum altitudinis quaſi infinitæ laminæ elaſticæ inducit. Sit AMB figura hæc quæ quæritur, & poſito AP $= x$, PM $= y$, AM $= s$; elementum Mm in directione normali MN urgebitur vi ipſi ds proportionali; unde erit $dv = n\,ds$, & $dt = 0$. Hinc orietur vis verticalis $dp = n\,dx$, & horizontalis $dq = -n\,dy$; ex quibus ſtatim fit $p = nx$ & $q = -ny$; ideoque in æquatione prima fiet $\frac{Ekk}{R} = P(c + x) - Qy + \frac{1}{2}nxx + \frac{1}{2}nyy$. *Fig. 19.*

Coordinatæ vero x & y ita quantitatibus conſtantibus augeri diminuive poſſunt, ut æquatio pro curva hujuſmodi faciem acquirat $xx + yy = A + \frac{B}{R}$. Hæc autem æquatio ſi multiplicetur per $xdx + ydy$, fiet integrabilis; eſt enim $\int \frac{xdx + ydy}{R} = -\int \frac{x + y\omega}{(1+\omega\omega)^{3:2}} d\omega$, [poſito $dy = \omega dx$] $= \frac{y - \omega x}{\surd(1+\omega\omega)} = \frac{ydx - xdy}{ds}$. Hanc ob rem, poſt integrationem conſtantibus mutatis, prodibit $(xx+yy)^2 = A(xx+yy) + \frac{B(ydx - xdy)}{ds} + C$. Sit $\surd(xx+yy) = z$ & $y = uz$; erit $x = z\surd(1 - uu)$; unde $ydx - xdy = \frac{-zzdu}{\surd(1-uu)}$, & $ds = \surd(dz^2 + \frac{zzdu^2}{1-uu})$. Ergo poſito $\frac{du}{\surd(1-uu)} = dr$, erit $z^4 - Az^2 - C = \frac{-Bzzdr}{\surd(dz^2 + zzdr^2)}$; hincque $dr = \frac{du}{\surd(1-uu)} = \frac{dz(z^4 - Az^2 - C)}{z\surd(B^2zz - (z^4 - Az^2 - C)^2)}$. Curva hæc ergo ſi fuerit $A = 0$ & $C = 0$, erit algebraica; habebitur enim hæc æquatio $\frac{du}{\surd(1-uu)} = \frac{zzdz}{\surd(B^2 - z^6)} = \frac{3zzdz}{3\surd(a^6 - z^6)}$, quæ integrata dat A ſin. $u = \frac{1}{3}$ A ſin. $\frac{z^3}{a^3}$, ſeu $\frac{z^3}{a^3} = 3u - 4u^3 = \frac{3y}{z} - \frac{4y^3}{z^3}$; unde hæc reſultat æquatio $z^6 = 3a^3yzz - 4a^3y^3$; ſeu, ob $zz = xx + yy$, hæc $x^6 + 3x^4y^2 + 3xxy^4 + y^6 = 3a^3xxy - a^3y^3$.

De motu oſcillatorio laminarum elaſticarum.

63. Ex his etiam motus oſcillatorius laminarum elaſticarum utcunque ad motum comparatarum definiri poteſt: quod argumentum profecto digniſſimum primum excolere cœpit Vir Celeb. *Daniel* BERNOULLI, mihique, jam ante complures annos, Problema de oſcillationibus laminæ elaſticæ altero termino parieti firmo infixæ determinandis propoſuit, cujus Solutionem exhibui

Digitized by Google

exhibui in *Comment. Petropol.* Tomo VII. Ex hoc autem tempore, cum mihi commodius hoc Problema tractare contigit; tum etiam per commercium cum Celeb. BERNOULLIO plures acceſſerunt aliæ quæſtiones & conſiderationes, quarum enodationem, ob materiæ affinitatem, hic adjungam. Quando autem motus vibratorius eſt ſatis promtus, tum ſimul a lamina vibrante ſonus editur, cujus tenor ac relatio ad alios, ope doctrinæ de ſonis, ex his principiis determinabitur. Et quoniam ſonorum indoles facillime ad experimenta revocatur; hoc ipſo conſenſus calculi cum veritate explorari, atque adeo Theoria confirmari poterit: quo pacto, cognitio noſtra circa naturam corporum elaſticorum non parum amplificabitur.

64. Primum autem monendum eſt, hîc tantum circa oſcillationes minimas quæſtionem inſtitui; atque adeo intervallum, per quod lamina inter oſcillandum excurrit, eſſe quaſi infinite parvum. Neque vero, hac limitatione, uſus & applicatio quicquam diminuitur: non ſolum enim oſcillationes, ſi per majora ſpatia fierent, iſochroniſmo deſtituerentur; ſed etiam ſonorum diſtinctorum formatio, ad quam hic potiſſimum ſpectamus, minimas oſcillationes requirit. Conſidero igitur hic primum laminam elaſticam uniformem naturaliter rectam, cujus alter terminus B pavimento immobili firmiter ſit infixus, ita ut lamina ſibi relicta ſitum teneat rectum BA. Sit hujus laminæ longitudo $AB = a$, ejuſque elaſticitas abſoluta in ſingulis locis $= Ekk$; ab ejus vero pondere vel mentem revocamus, vel infixionem ejuſmodi ſtatuimus, ut ejus ſtatus a gravitate turbari nequeat. *Fig. 20.*

65. Jam lamina hæc a vi quacunque impulſa vibrationes peragat minimas, circa ſtatum naturalem BA utrinque excurrendo per minima intervalla Aa. Sitque BMa ſtatus quiſpiam, quem lamina inter oſcillandum tenet; qui quoniam infinite parum tantum diſtat a ſtatu naturali BPA, rectæ MP, Aa ſimul repræſentabunt vias, quas laminæ puncta M & a percurrunt, vel potius hæ rectæ ad vias veras rationem habebunt a ratione æqualitatis infinite parum diſcrepantem. Ad motum autem oſcillatorium determinandum, abſolute neceſſe eſt naturam *De oſcillationibus laminæ elaſticæ altero termino muro infixæ.*

 curvæ

Digitized by Google

curvæ BMa, quam lamina inter oſcillandum induit, noſſe. Sit igitur AP $= x$, PM $= y$, arcus aM $= s$, & radius oſculi in M $= R$; & intervallum minimum Aa $= b$; atque, ex conditione memorata, erit arcus s proxime æqualis abſciſſæ x, ac proinde pro ds ſumi poterit dx: præ dx enim evaneſcet dy. Et cum, poſito dx conſtante, ſit generatim radius oſculi $= \frac{ds^3}{dx\,ddy}$, erit præſenti caſu $R = \frac{dx^2}{ddy}$; nam curva BMa convexitatem axi BA obvertit, & quia lamina in B muro firmiter eſt infixa, erit recta AB tangens curvæ in puncto B.

66. His poſitis, tam ad naturam curvæ BMa quam ad ipſum motum oſcillatorium determinandum, ſit f longitudo penduli ſimplicis iſochroni: oſcillationes enim minimas eſſe iſochronas, cum natura rei declarat, tum ipſe calculus inſtituendus monſtrabit. Acceleratio ergo, qua laminæ punctum M verſus P urgetur, erit $= \frac{PM}{f} = \frac{y}{f}$. Quare ſi maſſa totius laminæ ponatur $= M$, quæ per ejus pondus exprimitur; erit elementi Mm $= ds = dx$ maſſa $= \frac{M\,dx}{a}$; unde vis motrix elementum Mm in directione MP ſollicitans erit $= \frac{My\,dx}{af}$; ſicque vires, quibus ſingulæ laminæ particulæ ad motum actu cientur, innoteſcunt, cum ex ipſa curva BMa, tum ex longitudine penduli ſimplicis iſochroni f. Quoniam vero lamina a vi elaſtica revera ad motum incitatur; ex hac cognita viciſſim & natura curvæ BMa, & longitudo penduli ſimplicis iſochroni determinabitur.

67. Quoniam ergo lamina perinde movetur, ac ſi ſingulis ipſius elementis Mm in directione MP vires eſſent applicatæ $= \frac{My\,dx}{af}$; ſequitur, ſi laminæ ſingulis elementis Mm in directionibus contrariis Mπ æquales vires $\frac{My\,dx}{af}$ applicarentur, laminam in ſtatu BMa æquilibrari. Hinc lamina inter oſcillandum eandem curvaturam ſubibit, quam indueret quieta, ſi in ſingu-

ſingulis punctis M ſollicitaretur viribus $\frac{My\,dx}{af}$ in directionibus Mπ. Per regulam ergo ſupra §. 56 inventam, colligantur omnes hæ vires per arcum aM applicatæ, atque prodibit ſumma $= \frac{M}{af}\int y\,dx$, quæ ibi in locum ipſius p ſubſtitui debet. Quare cum vires reliquæ P, Q, & q, quæ ibi habebantur, evaneſcant, natura curvæ exprimetur æquatione $\frac{Ekk}{R} = \int p\,dx$: unde habebitur $\frac{Ekk}{R} = \frac{M}{af}\int dx \int y\,dx$. Cum vero ſit $R = \frac{dx^2}{ddy}$, erit $\frac{Ekk\,ddy}{dx^2} = \frac{M}{af}\int dx \int y\,dx$; & differentiando $\frac{Ekk\,d^3y}{dx^2} = \frac{M\,dx}{af} \times \int y\,dx$: denuoque differentiando prodibit iſta æquatio differentialis quarti ordinis. $Ekk\,d^4y = \frac{M\,y\,dx^4}{af}$.

68. Hac ergo æquatione & natura curvæ BMa exprimitur, & ex eadem, ſi ad caſum oblatum accommodetur, longitudo f determinabitur; qua cognita, ipſe motus oſcillatorius innoteſcet. Ante omnia autem hanc æquationem integrari oportet: quæ cum pertineat ad id æquationum differentialium altiorum graduum genus, cujus integrationem generalem exhibui in *Miſcell. Berol.* Volumine VII, hinc ſequens æquatio integralis reperietur; ponendo brevitatis ergo $\frac{Ekk.\,af}{M} = c^4$; prodibit ſcilicet

$$y = A\,e^{\frac{x}{c}} + B\,e^{-\frac{x}{c}} + C\,\text{ſin.}\frac{x}{c} + D\,\text{coſ.}\frac{x}{c},$$

ubi e denotat numerum cujus logarithmus hyperbolicus eſt $= 1$; & ſin. $\frac{x}{c}$ & coſ. $\frac{x}{c}$ denotant ſinum & coſinum arcus $= \frac{x}{c}$ in circulo, cujus radius $= 1$, aſſumpti. Tum vero $A, B, C,$ & D ſunt quatuor conſtantes arbitrariæ per quadruplicem integrationem introductæ, quas ex accommodatione calculi ad præſentem caſum definire oportet.

69. Determinatio autem conſtantium ſequenti modo inſtituetur.

tur. Primum poſito $x = 0$, fieri debet $y = b$; hinc ergo oritur iſta æquatio, $b = A + B + D$, quæ eſt *prima*.

Secundo, cum ſit $\frac{c^4 ddy}{dx^2} = \int dx \int y dx$; facto $x = 0$, fieri debet $\frac{ddy}{dx^2} = 0$; at eſt $\frac{ddy}{dx^2} = \frac{A}{cc} e^{\frac{x}{c}} + \frac{B}{cc} e^{-\frac{x}{c}} - \frac{C}{cc} \text{ſin.} \frac{x}{c} - \frac{D}{cc} \text{coſ.} \frac{x}{cc}$: unde oritur hæc æquatio *ſecunda*, $0 = A + B - D$.

Tertio, cum ſit $\frac{c^4 d^3 y}{dx^3} = \int y dx$; poſito $x = 0$, ſimul $\frac{d^3 y}{dx^3}$ evaneſcere debet: quia ergo erit $\frac{c^3 d^3 y}{dx^3} = A e^{\frac{x}{c}} - B e^{-\frac{x}{c}} - C \text{coſ.} \frac{x}{c} + D \text{ſin.} \frac{x}{c}$: prodit æquatio *tertia*, $0 = A - B - C$.

Quarto autem, ſi ponatur $x = a$, applicata y evaneſcit, unde obtinebitur æquatio *quarta*, $0 = A e^{\frac{a}{c}} + B e^{-\frac{a}{c}} + C \text{ſin.} \frac{a}{c} + D \text{coſ.} \frac{a}{c}$.

Quinto, quia AB eſt tangens curvæ in puncto B; facto $x = a$, fieri debet $\frac{dy}{dx} = 0$: unde prodit æquatio *quinta*, $0 = A e^{\frac{a}{c}} - B e^{-\frac{a}{c}} + C \text{coſ.} \frac{a}{c} - D \text{ſin.} \frac{a}{c}$.

Ex his ergo quinque æquationibus, primum quatuor conſtantes A, B, C, D definientur; tum vero, in quo cardo rei verſatur, determinabitur valor ipſius $c = \sqrt[4]{\frac{Ekk.af}{M}}$; ex quo longitudo penduli ſimplicis iſochroni f elicietur; quo ipſo, durationes oſcillationum cognoſcentur.

70. Ex æquationibus ſecunda & tertia, conſtantes C & D ex A & B ita definientur, ut ſit

$$C = A - B, \quad \& \quad D = A + B.$$

qui

qui valores in æquationibus quarta & quinta ſubſtituti dabunt

$$0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} + (A-B)\,\text{ſin.}\,\frac{a}{c} + (A+B)\,\text{coſ.}\,\frac{a}{c}$$

$$0 = Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} + (A-B)\,\text{coſ.}\,\frac{a}{c} - (A+B)\,\text{ſin.}\,\frac{a}{c}\,;$$

ex quibus eruitur.

$$\frac{A}{B} = \frac{-e^{-\frac{a}{c}} + \text{ſin.}\,\frac{a}{c} - \text{coſ.}\,\frac{a}{c}}{e^{\frac{a}{c}} + \text{ſin.}\,\frac{a}{c} + \text{coſ.}\,\frac{a}{c}} = \frac{e^{-\frac{a}{c}} + \text{coſ.}\,\frac{a}{c} + \text{ſin.}\,\frac{a}{c}}{e^{\frac{a}{c}} + \text{coſ.}\,\frac{a}{c} - \text{ſin.}\,\frac{a}{c}}$$

unde obtinebitur hæc æquatio,

$$0 = 2 + (e^{\frac{a}{c}} + e^{-\frac{a}{c}})\,\text{coſ.}\,\frac{a}{c},$$

ſeu $e^{\frac{2a}{c}}\,\text{coſ.}\,\frac{a}{c} + 2e^{\frac{a}{c}} + \text{coſ.}\,\frac{a}{c} = 0$,

quæ dat $e^{\frac{a}{c}} = \dfrac{-1 \pm \text{ſin.}\,\frac{a}{c}}{\text{coſ.}\,\frac{a}{c}}$. Cum igitur $e^{\frac{a}{c}}$ ſit quantitas affirmativa, coſinus anguli $\frac{a}{c}$ erit negativus; ideoque angulus $\frac{a}{c}$ recto major.

71. Ex hac æquatione intelligitur dari infinitos angulos $\frac{a}{c}$ quæſito ſatisfacientes, ex quibus infiniti diverſi modi oſcillationum ejuſdem laminæ oriuntur. Curva enim in uno pluribusve punctis axem A B ſecare poteſt, antequam in B axem tangat; ex quo ejuſdem laminæ plures, imo infiniti, oſcillandi modi æque ſunt poſſibiles. Cum igitur hîc inprimis contemplemur caſum, quo B primum eſt punctum, ubi lamina ab axe A B tangitur; huic caſui ſatisfaciet minimus angulus $\frac{a}{c}$ æquationem inven-

Digitized by Google

inventam reſolvens; qui angulus cum ſit recto major, ponatur $\frac{a}{c} = \frac{1}{2}\pi + \phi$; exiſtente ϕ angulo recto minore. Hinc, ob ſin. $\frac{a}{c} = $ coſ. ϕ, & coſ. $\frac{a}{c} = -$ ſin. ϕ, obtinebitur duplex æquatio.

$$e^{\frac{a}{c}} = \frac{1 \mp \text{coſ. } \phi}{\text{ſin. } \phi}.$$

quæ præbet, vel $e^{\frac{a}{c}} = $ tang. $\frac{1}{2}\phi$, vel $e^{\frac{a}{c}} = $ cot. $\frac{1}{2}\phi$, quarum poſterior minorem dabit valorem pro angulo ϕ; quæ ergo ad caſum propoſitum erit accommodata.

72. Sequentes poſſibiles oſcillationum modi reperientur, ſi pro $\frac{a}{c}$ ponantur anguli duobis rectis majores, tribus vero minores. Sic poſito $\frac{a}{c} = \frac{3}{2}\pi - \phi$, erit ſin. $\frac{a}{c} = -$ coſ. ϕ, & coſ. $\frac{a}{c} = -$ ſin. ϕ; unde fit $e^{\frac{a}{c}} = \frac{1 + \text{coſ. } \phi}{\text{ſin. } \phi}$, ſeu, vel $e^{\frac{a}{c}} = $ tang. $\frac{1}{2}\phi$, vel $e^{\frac{a}{c}} = $ cot. $\frac{1}{2}\phi$. Simili modo alii oſcillationum modi reperientur, ponendo $\frac{a}{c} = \frac{5}{2}\pi + \phi$; $\frac{a}{c} = \frac{7}{2}\pi - \phi$, &c. Ex quibus omnibus, ſi ſumantur logarithmi hyperbolici, orientur ſequentes æquationes:

I. $\frac{1}{2}\pi + \phi = l$ cot. $\frac{1}{2}\phi$; II. $\frac{1}{2}\pi + \phi = l$ tang. $\frac{1}{2}\phi$;
III. $\frac{3}{2}\pi - \phi = l$ cot. $\frac{1}{2}\phi$; IV. $\frac{3}{2}\pi - \phi = l$ tang. $\frac{1}{2}\phi$;
V. $\frac{5}{2}\pi + \phi = l$ cot. $\frac{1}{2}\phi$; VI. $\frac{5}{2}\pi + \phi = l$ tang. $\frac{1}{2}\phi$;
VII. $\frac{7}{2}\pi - \phi = l$ cot. $\frac{1}{2}\phi$; VIII. $\frac{7}{2}\pi - \phi = l$ tang. $\frac{1}{2}\phi$;
&c.

Harum autem æquationum tertia congruit cum ſecunda; poſito enim $\frac{1}{2}\phi = \frac{1}{2}\pi - \frac{1}{2}\theta$, ut ſit cot. $\frac{1}{2}\phi = $ tang. $\frac{1}{2}\theta$; tertia tranſit in $\frac{1}{2}\pi + \theta = l$ tang. $\frac{1}{2}\theta$, quæ eſt ipſa ſecunda. Simili modo quarta congruit cum prima: tum quinta & octava inter ſe con-

Digitized by Google

congruunt; atque ſexta cum ſeptima. Quamobrem ſequentes tantum prodibunt æquationes diverſæ:

$$\begin{array}{ll} \text{I.} & \tfrac{1}{2}\pi + \phi = l\ \text{cot.}\ \tfrac{1}{2}\phi \\ \text{II.} & \tfrac{1}{2}\pi + \phi = l\ \text{tang.}\ \tfrac{1}{2}\phi \\ \text{III.} & \tfrac{5}{2}\pi + \phi = l\ \text{cot.}\ \tfrac{1}{2}\phi \\ \text{IV.} & \tfrac{5}{2}\pi + \phi = l\ \text{tang.}\ \tfrac{1}{2}\phi \\ \text{V.} & \tfrac{9}{2}\pi + \phi = l\ \text{cot.}\ \tfrac{1}{2}\phi \\ \text{VI.} & \tfrac{9}{2}\pi + \phi = l\ \text{tang.}\ \tfrac{1}{2}\phi \end{array}$$

&c.

72. Logarithmus autem hyperbolicus tangentis vel cotangentis cujuſpiam anguli reperitur, ſumendo logarithmum Tabularem, indeque auferendo logarithmum ſinus totius, atque reſiduum multiplicando per $2,302585092994$; qui labor ut ſublevetur denuo logarithmis uti conveniet. Sit u logarithmus hyperbolicus tangentis ſeu cotangentis anguli $\frac{1}{2}\phi$, qui quæritur; ſumatur ex Tabulis logarithmus ejuſdem tangentis cotangentiſve, qui logarithmo ſinus totius multatus ponatur $= v$. Cum ergo ſit $u = 2,302585092994 \times v$; erit, ſumendis logarithmis vulgaribus,

$$lu = lv + 0,3622156886.$$

Hoc logarithmo invento, cum ſit $u = \frac{n}{2}\pi + \phi$, erit $lu = l\left(\frac{n}{2}\pi + \phi\right)$. Ad hoc evolvendum, angulus ϕ in partibus radii exprimi debet, quemadmodum & π eodem modo exprimitur, dum eſt $\pi = 3,1415926535$, ac propterea . . . . $\frac{1}{2}\pi = 1,5707963267 9$. Angulus autem ϕ eodem modo exprimetur, ſi is in minuta ſecunda convertatur, atque ab hujus numeri logarithmo ſubtrahatur conſtanter $5,3144251332$; ſic enim prodibit $l\phi$, ex quo ad numeros regrediendo valor ipſius ϕ eruitur. Erit autem conſtanter pro unoquoque oſcillationum genere $\frac{a}{c} = u = \frac{n}{2}\pi + \phi$.

74. His circa calculum inſtituendum monitis, per approxi-

mationes valor anguli φ pro quovis oſcillationum genere non difficulter eruetur. Tribuendo enim pro lubitu ipſi φ valores aliquot, & per calculum determinando, & $\frac{n}{2}\pi + \varphi$, & $l\,\frac{\text{tang.}}{\text{cot.}}\,\frac{1}{2}\varphi$. mox valor ipſius φ prope verus cognoſcetur. Quod ſi autem habeantur limites anguli φ utcunque remoti, ſtatim invenientur limites propiores, ex hiſque tandem verus valor ipſius φ. Sic pro æquatione prima $\frac{a}{c} = \frac{1}{2}\pi + \varphi = l$ cot. $\frac{1}{2}\varphi$, ſequentes limites anguli φ erui $17°, 26'$, & $17°, 27'$, ex quibus per ſequentem calculum verus valor ipſius φ obtinebitur.

φ	=	17°, 26′, 0″	17°, 27′, 0″
in min. ſec.	=	62760″	62820″
log.	=	4, 7976829349	4, 7980979321
ſubtr.		5, 3144251332	5, 3144251332
$l\,\varphi$	=	9, 4832578017	9, 4836727989
φ	=	0, 3042690662	0, 3045599545
$\frac{1}{2}\pi$	=	1, 5707963268	1, 5707963268
$\frac{1}{2}\pi + \varphi$	=	1, 8750653930	1, 8753562813
$\frac{1}{2}\varphi$	=	8°, 43′, 0″	8°, 43′, 30″
l cot. $\frac{1}{2}\varphi$	=	10, 8144034109	10, 8139819342
v	=	0, 8144034109	0, 8139819342
$l\,v$	=	9, 9108395839	9, 9106147660
add.	=	0, 3622156886	0, 3622156886
$l\,u$	=	0, 2730552725	0, 2728304546
u	=	1, 8752331540	1, 8742626675
diff.		+ 1677610	— 10936138

Ex his ergo utriuſque limitis erroribus concluditur fore $\varphi = 17°, 26', 7''\,\frac{98}{100}$, & $\frac{1}{2}\pi + \varphi$, ſeu $\frac{a}{c} = 107°, 26', 7''\,\frac{98}{100}$. Cum vero in minutis ſecundis ſit $\varphi = 62767{,}98$, erit

$l\,\varphi =$

$$
\begin{array}{rcl}
l\,\Phi & = & 4{,}7977381525 \\
\text{ſubtr.} & = & \underline{5{,}3144251332} \\
 & & 9{,}4833130193 \\
\text{ergo } \Phi & = & 0{,}3043077545 \\
\text{add. } \tfrac{1}{2}\pi & = & \underline{1{,}5707963268} \\
\frac{a}{c} & = & 1{,}8751040813
\end{array}
$$

quo invento, erit $\frac{A}{B} = \text{tang.}\ \tfrac{1}{2}\Phi = 0{,}1533390624$. Reperitur ergo ratio conſtantium A & B, ex quibus & ratio reliquarum conſtantium C & D ad illas cognoſcetur.

75. Reſtat adhuc prima æquatio $b = A + B + D$, quæ, ob $D = A + B$, abit in $b = 2A + 2B$; ideoque $A + B = \tfrac{1}{2}b$: cum ergo ſit $\frac{A}{B} = \text{tang.}\ \tfrac{1}{2}\Phi$, fiet $B\,(1 + \text{tang.}\ \tfrac{1}{2}\Phi) = \tfrac{1}{2}b$, & $B = \frac{b}{2 + 2\ \text{tang.}\ \frac{1}{2}\Phi}$. Unde ex tang. $\tfrac{1}{2}\Phi = 0{,}1533390624$ ſingulæ æquationis conſtantes ſequenti modo determinabuntur:

$$\frac{A}{b} = \frac{\text{tang.}\ \frac{1}{2}\Phi}{2\,(1 + \text{tang.}\ \frac{1}{2}\Phi)} = \frac{0{,}1533390624}{2{,}3066781248}$$

$$\frac{B}{b} = \frac{1}{2\,(1 + \text{tang.}\ \frac{1}{2}\Phi)} = \frac{1{,}0000000000}{2{,}3066781248}$$

$$\frac{C}{b} = \frac{-1 + \text{tang.}\ \frac{1}{2}\Phi}{2\,(1 + \text{tang.}\ \frac{1}{2}\Phi)} = \frac{-0{,}8466609376}{2{,}3066781248}$$

$$\frac{D}{b} = \frac{1 + \text{tang.}\ \frac{1}{2}\Phi}{2\,(1 + \text{tang.}\ \frac{1}{2}\Phi)} = \frac{1{,}1533390624}{2{,}3066781248}$$

quibus inventis, natura curvæ aMB, quam lamina inter oſcillandum induit, hac exprimetur æquatione

$$\frac{y}{b} = \frac{A}{b}\,e^{\frac{x}{c}} + \frac{B}{b}\,e^{-\frac{x}{c}} + \frac{C}{b}\,\text{ſin.}\ \frac{x}{c} + \frac{D}{b}\ \text{coſ.}\ \frac{x}{c}.$$

76. Quod autem ad oſcillationum velocitatem attinet, ea

 ex

ex æquatione $\frac{a}{c} = 1,8751040813$ cognoſcetur. Ponatur brevitatis gratia: $n = 1,8751040813$ ut ſit $a = nc$.

Et cum ſit $c^4 = \frac{Ekk.af}{M}$, ubi $\frac{M}{a}$ exprimit gravitatem ſpecificam laminæ, & Ekk elaſticitatem abſolutam; eo modo, quo hactenus ſum uſus, erit $a^4 = n^4 . Ekk . \frac{a}{M} . f$, ideoque $f = \frac{a^4}{n^4} . \frac{1}{Ekk} . \frac{M}{a}$, ex quo longitudo penduli ſimplicis iſochroni tenebit rationem compoſitam ex quadruplicata longitudinis laminæ, ſimplici gravitatis ſpecificæ, & inverſa elaſticitatis abſolutæ. Sit g longitudo penduli ſimplicis ſingulis minutis ſecundis oſcillantis, ita ut ſit $g = 3,16625$ ped. Rhenani; quia durationes oſcillationum ſunt in ſubduplicata ratione pendulorum, tempus unius oſcillationis a lamina noſtra elaſtica factæ, erit $= \frac{\sqrt{f}}{\sqrt{g}}$ ſecund. $= \frac{aa}{nn} \sqrt{\frac{1}{g} . \frac{1}{Ekk} . \frac{M}{a}}$; unde numerus oſcillationum uno minuto ſecundo editarum erit $= \frac{nn}{aa} \sqrt{g . Ekk . \frac{a}{M}}$, qui numerus exprimit ſoni quem lamina excitat tenorem. Soni ergo a diverſis laminis elaſticis uno termino muro infixis editi erunt in ratione compoſita ſubduplicata elaſticitatum abſolutarum directe, inverſa ſubduplicata gravitatum ſpecificarum, & inverſa duplicata longitudinum. Quare ſi duæ laminæ elaſticæ tantum longitudine differant, erunt ſoni reciproce ut quadrata longitudinum; ſcilicet lamina duplo longior edet ſonum duabus octavis graviorem. Corda autem tenſa duplo longior ſonum una tantum octava graviorem edit, ſi tenſio maneat eadem. Ex quo patet ſonos laminarum elaſticarum longe aliam ſequi rationem, atque ſonos cordarum tenſarum.

77. Quod ad naturam curvæ aMB ultra terminos a & B continuatæ attinet, primum quidem patet curvam ultra a divergendo ab axe BA continuato progredi. Poſito enim x negativo fiet

$$y = B\,e^{\frac{x}{c}} + A\,e^{-\frac{x}{c}} - C\,\text{ſin.}\,\frac{x}{c} + D\,\text{coſ.}\,\frac{x}{c}.$$

Hic jam omnes termini ſunt affirmativi, quia ſolus coefficiens C ante obtinuerat valorem negativum; unde dum creſcit x, etiam y creſcere debet, quia numerus B major eſt quam A; atque adeo terminus $B\,e^{\frac{x}{c}}$ prævalet. Quam primum autem $\frac{x}{c}$ valorem ſaltem mediocrem eſt adeptum; tum iſte terminus $B\,e^{\frac{x}{c}}$ tantopere creſcit, ut reliqui termini præ eo quaſi evaneſcant. Ob eandem rationem, quia curvæ in B radius oſculi non eſt $= \infty$; eſt enim $\frac{Ekk}{R} = \frac{M}{af}\int dx \int y\,dx$; curva in B non habebit punctum flexus contrarii, ideoque ad eandem axis AB partem ulterius progredietur; aucta autem abſciſſa x ultra $AB = a$, tum primus terminus $A\,e^{\frac{x}{c}}$ mox tam fit magnus, ut reliqui præ eo pro nihilo reputari queant.

78. Hic igitur eſt primus oſcillationum modus inter illos innumerabiles, ad quos eadem lamina ſe componere poteſt. Secundus modus in figura repræſentatus quo lamina in B fixa axem AB in uno puncto O trajicit, deducetur ex æquatione $\frac{a}{c} = \frac{1}{2}\pi + \Phi = l\,\text{tang.}\,\frac{1}{2}\Phi$, ſeu hac $\frac{3}{2}\pi - \Phi = l\,\text{cot.}\,\frac{1}{2}\Phi = \frac{a}{c}$. Fig. 21.
Hic per nonnulla tentamina inveni angulum Φ contineri intra hos limites, $1°, 2', 40''$ & $1°, 3', 0''$, ex quibus ut ante verus valor ipſius Φ eruetur.

$\Phi =$	$1^\circ, 2', 40''$	$1^\circ, 3', 0''$
in min. ſec. $=$	$3760''$	$3780''$
log. $=$	3,5751878450	3,5774917998
ſubtr. $=$	5,3144251332	5,3144251332
$l\Phi =$	8,2607627118	8,2630666666
$\Phi =$	0,0182289944	0,0183259571
$\frac{3}{2}\pi =$	4,7123889804	4,7123889804
$\frac{a}{c} =$	4,6941599860	4,6940630233
$\frac{1}{2}\Phi =$	$31', 20''$	$31', 30''$
l cot. $\frac{1}{2}\Phi =$	2,0402552577	2,0379511745
$lv =$	0,3096845055	0,3091937748
add.	0,3622156886	0,3622156886
$lu =$	0,6719001941	0,6714094634
$u =$	4,6978613391	4,6925559924
$\frac{a}{c} =$	4,6941599860	4,6940630233
Error	$+ 37013531$	$- 15070309$

Ex his erroribus concluditur verus valor anguli $\Phi = 1^\circ, 2', 54'' \frac{213}{1000}$, & $\frac{a}{c} = 268^\circ, 57', 5'' \frac{787}{1000}$. Cum igitur ſit $\Phi = 3774,213''$ erit

$$
\begin{aligned}
l\Phi &= 3,5768264061 \\
\text{ſubtr.} &= 5,3144251332 \\
&\quad\; 8,2624012729 \\
\Phi &= 0,0182979009 \\
\text{a}\ \tfrac{3}{2}\pi &= 4,7123889804 \\
\frac{a}{c} &= 4,6940910795
\end{aligned}
$$

Sonus ergo laminæ priori modo oſcillantis erit ad ſonum ejuſdem laminæ hoc modo vibrantis, uti eſt quadratum numeri 1,8751040813 ad quadratum numeri 4,6940910795, hoc eſt ut

ut 1 ad 6,166891, ſeu in minimis numeris ut 4 ad 25, ſeu ut 1 ad $6\frac{4}{25}$. Unde ſonus poſterior erit ad priorem duplex octava cum quinta & cum hemitonio fere.

79. Pro ſequentibus oſcillationum modis ejuſdem laminæ elaſticæ, quibus lamina inter oſcillandum axem AB in duobus pluribuſve punctis interſecat, fit angulus Φ multo minor. Sic pro tertio modo habetur hæc æquatio $\frac{5}{2}\pi + \Phi = l\,\mathrm{cot.}\,\frac{1}{2}\Phi = \frac{a}{c}$. Cum ergo ſit $e^{\frac{5}{2}\pi + \Phi} = \mathrm{cot.}\,\frac{1}{2}\Phi$, ob Φ angulum vehementer parvum, erit $e^{\frac{5}{2}\pi + \Phi} = e^{\frac{5}{2}\pi}(1 + \Phi + \frac{1}{2}\Phi^2 + \frac{1}{6}\Phi^3 + \&c.)$ & cot. $\frac{1}{2}\Phi = \frac{1 - \frac{1}{8}\Phi\Phi}{\frac{1}{2}\Phi - \frac{1}{48}\Phi^3} = \frac{2}{\Phi} - \frac{\Phi}{6}$. Hinc erit proxime $e^{\frac{5}{2}\pi} = \frac{2}{\Phi}$; ideoque $\Phi = 2e^{-\frac{5}{2}\pi}$, & propius $\Phi = \frac{1}{1 + \frac{1}{2}e^{\frac{5}{2}\pi}}$; unde erit $\frac{a}{c} = \frac{5}{2}\pi + \frac{2}{e^{\frac{5}{2}\pi} + 2}$; qui poſterior terminus eſt quàm-minimus. Simili modo pro quarto oſcillationum modo, erit proxime $\frac{a}{c} = \frac{7}{2}\pi - 2e^{-\frac{7}{2}\pi}$, & ita porro: ob hos alteros terminos evaneſcentes, ipſius $\frac{a}{c}$ valores erunt $\frac{9}{2}\pi$, $\frac{11}{2}\pi$, &c. qui eo minus a veritate aberrabunt, quo ulterius progredientur.

80. Conſideremus jam laminam elaſticam nuſquam fixam; ſed liberam vel plano politiſſimo incumbentem, vel remota gravitate, in ſpatio vacuo verſantem. Facile autem patet hujuſmodi laminam motum oſcillatorium recipere poſſe, dum lamina acb ſeſe incurvando alternatim cis & ultra ſtatum quietis AB excurrit. Motus igitur iſte oſcillatorius ſimili modo, quo in caſu præcedente, definiri poterit, dummodo calculus debito modo ad hunc caſum accommodetur. Sit igitur acb figura laminæ incurvata quam inter oſcillandum obtinet, at ACB ſit ſitus ejuſdem laminæ in ſtatu æquilibrii, per quem in quavis oſcillatione tranſit. Ponatur, ut ante, longitudo laminæ AB $= a$, ejus elaſticitas abſoluta $= Ekk$, atque pondus ſeu maſſa $= M$. Deinde

De oſcillationibus laminæ elaſticæ liberæ. *Fig. 22.*

Deinde ſit abſciſſa AP $= x$, applicata PM $= y$, arcus aM $= s$, qui cum abſciſſa x confundetur; ita ut ſtatui queat $ds = dx$; ex quo radius oſculi in M orietur $= \frac{dy^2}{ddy} = R$. Sit autem porro applicata prima Aa $= b$. His poſitis, ratiocinium ut ante inſtituendo, ad eandem pervenietur æquationem $\frac{Ekk}{R} = \frac{M}{af} \int dx \int y dx = \frac{Ekk ddy}{dx^2}$.

81. Si igitur ponamus $\frac{Ekk \cdot af}{M} = c^4$, ubi f ut ante exprimit longitudinem penduli ſimplicis iſochroni; habebitur, integrando, pro curva hæc æquatio

$$y = Ae^{\frac{x}{c}} + Be^{-\frac{x}{c}} + C\,\text{ſin.}\,\frac{x}{c} + D\,\text{coſ.}\,\frac{x}{c};$$

quæ ad præſentem caſum ita accommodabitur. Primo, ſi ponatur $x = 0$; fieri debet $y = b$; unde fit

$$b = A + B + D.$$

Secundo, cum ſit $\frac{c^4 ddy}{dx^2} = \int dx \int y dx$; poſito $x = 0$; fieri debet $\frac{ddy}{dx^2} = 0$, unde prodit

$$0 = A + B - D.$$

Tertio, cum ſit $\frac{c^4 d^3y}{dx^3} = \int y dx$, poſito $x = 0$; fieri quoque debet $\frac{d^3y}{dx^3} = 0$, unde naſcitur:

$$0 = A - B - D.$$

Quarto, ſi ponatur $x = a$, evaneſcere debet $\int y dx$, ſeu $\frac{d^3y}{dx^3}$; propterea quod $\int y dx$ exprimit ſummam omnium virium laminam in directione ad axem AB normali trahentium, quæ ſumma ſi non eſſet $= 0$, ipſa lamina motu locali promoveretur, contra inſtitutum; erit ergo, ob hanc rationem,

$$0 = Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} - C\,\text{coſ.}\,\frac{a}{c} + D\,\text{ſin.}\,\frac{a}{c}.$$

Quinto

Quinto; quia lamina in extremitate B eſt libera, ibi curvaturam nullam habere poterit; eritque ideo, poſito $x = a$, quoque $\frac{ddy}{dx^2} = 0$; unde erit

$$0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} - C\,\text{ſin.}\,\frac{a}{c} - D\,\text{coſ.}\,\frac{a}{c}.$$

His igitur quinque conditionibus in computum ductis, non ſolum quatuor conſtantes A, B, C & D determinabuntur; ſed etiam fractionis $\frac{a}{c}$ valor reperietur; ex quo proinde longitudo penduli ſimplicis iſochroni f innoteſcet.

82. Ex harum æquationum ſecunda & tertia, obtinetur $D = A + B$, & $C = A - B$, qui in ſequentibus ſubſtituti præbebunt

$$0 = Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} - (A - B)\,\text{coſ.}\,\frac{a}{c} + (A + B)\,\text{ſin.}\,\frac{a}{c}$$

$$0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} - (A - B)\,\text{ſin.}\,\frac{a}{c} - (A + B)\,\text{coſ.}\,\frac{a}{c};$$

ex quibus reperitur:

$$\frac{A}{B} = \frac{e^{-\frac{a}{c}} - \text{coſ.}\,\frac{a}{c} - \text{ſin.}\,\frac{a}{c}}{e^{\frac{a}{c}} - \text{coſ.}\,\frac{a}{c} + \text{ſin.}\,\frac{a}{c}} = \frac{-e^{-\frac{a}{c}} - \text{ſin.}\,\frac{a}{c} + \text{coſ.}\,\frac{a}{c}}{e^{\frac{a}{c}} - \text{ſin.}\,\frac{a}{c} - \text{coſ.}\,\frac{a}{c}};$$

ex qua æqualitate elicitur iſta æquatio

$$0 = 2 - e^{\frac{a}{c}}\,\text{coſ.}\,\frac{a}{c} - e^{-\frac{a}{c}}\,\text{coſ.}\,\frac{a}{c};\ \text{ſeu}\ e^{\frac{a}{c}} = \frac{1 \pm \text{ſin.}\,\frac{a}{c}}{\text{coſ.}\,\frac{a}{c}}$$

unde ſequentes formabuntur æquationes

I. $\frac{a}{c} = \frac{1}{2}\pi - \Phi = l$ tang. $\frac{1}{2}\Phi$, quæ dat $\frac{a}{c} = 0$.

pro ſitu laminæ naturali

II. $\frac{a}{c} = \frac{1}{2}\pi - \Phi = l \cot. \frac{1}{2}\Phi$

III. $\frac{a}{c} = \frac{3}{2}\pi + \Phi = l \cot. \frac{1}{2}\Phi$

IV. $\frac{a}{c} = \frac{5}{2}\pi - \Phi = l \cot. \frac{1}{2}\Phi$

V. $\frac{a}{c} = \frac{7}{2}\pi + \Phi = l \cot. \frac{1}{2}\Phi$

VI. $\frac{a}{c} = \frac{9}{2}\pi - \Phi = l \cot. \frac{1}{2}\Phi$

&c.

83. Hæ æquationes iterum indicant innumerabiles oſcillationum modos, in quorum ſecundo lamina ſemel tantum axem AB interſecabit, in tertio bis, in quarto ter, in quinto quater, & ita porro. Ex quibus intelligitur modos ſecundum, quartum, ſextum &c. ad præſens inſtitutum non eſſe accommodatos. Quoniam enim in his numerus interſectionum eſt impar; laminæ ſitus inter oſcillandum in ſecundo foret talis, qualem Figura 23 repræſentat, in quo quamvis ſumma virium ſollicitantium per totam laminam evaneſcat, tamen ab iis lamina circa punctum medium C motum rotatorium acquireret: quia vires utrique ſemiſſi aC & bC applicatæ ad eundem laminæ motum rotatorium inducendum conſpirarent. Quam ob cauſam, cum omnino motus rotatorius excludi debeat, figura laminæ, quam inter oſcillandum induit, ita debet eſſe comparata, ut non ſolum
Fig. 22. virium ſollicitantium toti laminæ applicatarum ſit $= 0$, ſed etiam ut earum ſumma momentorum evaneſcat; quod obtinetur ſi curva in puncto medio c, diametro cC ſit prædita. Quod evenit ſi curva axem AB vel in duobus, vel in quatuor, vel generatim in punctorum numero pari ſecet; ex quo æquationes tertia, quinta, ſeptima &c. Solutiones tantum convenientes præbebunt.

84. Hæc ipſa limitatio in ipſa Problematis propoſitione con-

Digitized by Google

contenta reperietur, ſi ejuſmodi tantum curvas admittamus; quæ rectam C c habeant diametrum, ſeu in quibus valor ipſius y prodeat idem, ſi loco x ſcribatur $a - x$. Ponamus ergo in æquatione generali $a - x$ loco x: atque prodibit

$$y = Ae^{\frac{a}{c}}e^{-\frac{x}{c}} + Be^{-\frac{a}{c}}e^{\frac{x}{c}} + C\sin.\frac{a}{c}.\cos.\frac{x}{c} - C\cos.\frac{a}{c}.\sin.\frac{x}{c}$$
$$+ D\cos.\frac{a}{c}.\cos.\frac{x}{c} + D\sin.\frac{a}{c}.\sin.\frac{x}{c}$$

quæ cum congruere debeat cum æquatione

$$y = Ae^{\frac{x}{c}} + Be^{-\frac{x}{c}} + C\sin.\frac{x}{c} + D\cos.\frac{x}{c};$$

fiet $Ae^{\frac{a}{c}} = B$, $C(1 + \cos.\frac{a}{c}) = D\sin.\frac{a}{c}$, & $C\sin.\frac{a}{c} = D(1 - \cos.\frac{a}{c})$; quarum duæ poſteriores congruunt. Cum ergo ſit $\frac{A}{B} = e^{-\frac{a}{c}}$, hoc valore cum ſuperioribus comparato prodibit:

$$e^{-\frac{a}{c}} - \cos.\frac{a}{c} - \sin.\frac{a}{c} = 1 - e^{-\frac{a}{c}}\cos.\frac{a}{c} + e^{-\frac{a}{c}}\sin.\frac{a}{c}$$

ſeu $e^{-\frac{a}{c}} = \dfrac{1 + \cos.\frac{a}{c} + \sin.\frac{a}{c}}{1 + \cos.\frac{a}{c} - \sin.\frac{a}{c}} = \dfrac{1 + \sin.\frac{a}{c}}{\cos.\frac{a}{c}} = \dfrac{\cos.\frac{a}{c}}{1 - \sin.\frac{a}{c}}$.

85. Erit ergo $e^{\frac{a}{c}} = \dfrac{1 - \sin.\frac{a}{c}}{\cos.\frac{a}{c}}$: ſicque in æquatione prius inventa $e^{\frac{a}{c}} = \dfrac{1 \pm \sin.\frac{a}{c}}{\cos.\frac{a}{c}}$; ſemiſſis tantum ſ caſuum ſupra exhibitorum, ſcilicet ii qui ſunt numeris imparibus, præſens Problema reſolvent. Quare cum prima æquatio contineat

laminæ ſtatum naturalem, omnes oſcillationum modi in ſequentibus æquationibus continebuntur:

$$\text{I.}\ \frac{a}{c} = \tfrac{3}{2}\pi + \Phi = l\ \text{cot.}\ \tfrac{1}{2}\Phi$$

$$\text{II.}\ \frac{a}{c} = \tfrac{7}{2}\pi + \Phi = l\ \text{cot.}\ \tfrac{1}{2}\Phi$$

$$\text{III.}\ \frac{a}{c} = \tfrac{11}{2}\pi + \Phi = l\ \text{cot.}\ \tfrac{1}{2}\Phi$$

&c.

Æquationum ergo hàrum prima præbebit primum eumque principalem oſcillandi modum, pro quo valor anguli Φ ſimili modo, quo ſupra, per approximationem reperietur. Limites autem anguli Φ mox colliguntur eſſe 1°, 0′, 40″ & 1°, 1′, 0″, ex quibus per ſequentem calculum verus ipſius Φ valor eruitur.

$\Phi =$	1°, 0′, 40″	1°, 1′, 0″
ſeu	3640″	3660″
log. =	3,5611013836	3,5634810854
ſubtr. =	5,3144251332	5,3144251332
$l\,\Phi =$	8,2466762504	8,2490559522
$\Phi =$	0,0176472180	0,0177441807
$\frac{3}{2}\pi =$	4,7123889804	4,7123889804
$\frac{a}{c} =$	4,7300361984	4,7301331611
$\frac{1}{2}\Phi =$	30′, 20″	30′, 30″
$v =$	2,0543424742	2,0519626482
$lv =$	0,3126728453	0,3121694510
add.	0,3622156886	0,3622156886
$lu =$	0,6748885339	0,6743851396
$u =$	4,7302983543	4,7248186037
Error.	+ 636341	+ 53145574
		636341
	diff.	52509233

Hinc

Hinc intelligitur verum valorem ipſius Φ non intra iſtos limites contineri, ſed aliquantulum eſſe minorem quam $1^\circ, 0', 40''$. Nihilo vero minus is ex his erroribus reperietur. Sit enim $\Phi = 1^\circ, 0', 40'' - n''$; erit $20'' : 52509233 = n'' : 636341$; unde reperitur $n = \frac{2423}{10000}$, ita ut ſit

$$\Phi = 1^\circ, 0', 39 \frac{7576''}{10000}.$$

Cum ergo ſit $\Phi = 3639, 7576''$ erit

$l\Phi =$	3, 5610724615
ſubtr.	5, 3144251332
	8, 2466473283
$\Phi =$	0, 0176460428
$\frac{3}{2}\pi =$	4, 7123889804
$\frac{a}{c} =$	4, 7300350232.

86. Sit hic numerus $= m$, erit, ob $c^4 = \frac{Ekk.af}{M}$, $a^4 = \frac{m^4. Ekk. af}{M}$, & $f = \frac{a^4}{m^4}. \frac{1}{Ekk}. \frac{M}{a}$. Unde pari modo numerus oſcillationum ab hac lamina uno minuto ſecundo editarum, erit $= \frac{mm}{aa} \sqrt{g. Ekk. \frac{a}{M}}$, exiſtente $g = 3, 16625$ ped. Rhen. Quod ſi ergo eadem lamina, nunc altero termino B muro infixo, nunc libera ad ſonum edendum incitetur, erunt ſoni inter ſe ut nn ad mm, hoc eſt ut quadrata numerorum 1, 8751040813 & 4, 7300350232, hoc eſt ut 1 ad 6, 363236. Ratio ergo horum ſonorum erit proxime ut 11 ad 70: horum ergo ſonorum intervallum conſtituet duas octavas, cum quinta & hemitonio. Sin autem poſterior lamina libera duplo longior capiatur quam prior fixa, intervallum ſonorum erit fere ſexta minor.

87. Invento hoc valore fractionis $\frac{a}{c}$; æquatio pro curva,

quam lamina inter oſcillandum format, hactenus indeterminata poterit determinari. Cum enim ſit

$$e^{\frac{a}{c}} = \frac{1 - \text{ſin.}\frac{a}{c}}{\text{coſ.}\frac{a}{c}}, \text{ erit } B = \frac{1 - \text{ſin.}\frac{a}{c}}{\text{coſ.}\frac{a}{c}} A, \& \ C = A - B$$

$$= A\left(\text{coſ.}\frac{a}{c} + \text{ſin.}\frac{a}{c} - 1\right) : \text{coſ.}\frac{a}{c}, \& \ D = A + B$$

$$= A\left(\text{coſ.}\frac{a}{c} - \text{ſin.}\frac{a}{c} + 1\right) : \text{coſ.}\frac{a}{c}. \text{ Jam eſt } b = A + B$$

$$+ D = 2D = 2A\left(\text{coſ.}\frac{a}{c} - \text{ſin.}\frac{a}{c} + 1\right) : \text{coſ.}\frac{a}{c}; \text{ unde fit}$$

$$A = \frac{b\,\text{coſ.}\frac{a}{c}}{2\left(\text{coſ.}\frac{a}{c} - \text{ſin.}\frac{a}{c} + 1\right)} = \frac{b\left(1 + \text{ſin.}\frac{a}{c} - \text{coſ.}\frac{a}{c}\right)}{4\,\text{ſin.}\frac{a}{c}}$$

$$B = \frac{b\left(1 - \text{ſin.}\frac{a}{c}\right)}{2\left(\text{coſ.}\frac{a}{c} - \text{ſin.}\frac{a}{c} + 1\right)} = \frac{b\left(-1 + \text{ſin.}\frac{a}{c} + \text{coſ.}\frac{a}{c}\right)}{4\,\text{ſin.}\frac{a}{c}}$$

$$C = \frac{b\left(-1 + \text{ſin.}\frac{a}{c} + \text{coſ.}\frac{a}{c}\right)}{2\left(\text{coſ.}\frac{a}{c} - \text{ſin.}\frac{a}{c} + 1\right)} = \frac{b\left(1 - \text{coſ.}\frac{a}{c}\right)}{2\,\text{ſin.}\frac{a}{c}}.$$

$$D = \frac{b}{2} = \frac{b\,\text{ſin.}\frac{a}{c}}{2\,\text{ſin.}\frac{a}{c}}.$$

His ſubſtitutis oritur hæc æquatio: $\frac{y}{b} =$

$$\frac{e^{\frac{x}{c}}\,\text{coſ.}\frac{a}{c} + e^{-\frac{x}{c}}\left(1 - \text{ſin.}\frac{a}{c}\right)}{2\left(1 - \text{ſin.}\frac{a}{c} + \text{coſ.}\frac{a}{c}\right)} + \frac{\left(1 - \text{coſ.}\frac{a}{c}\right)\text{ſin.}\frac{x}{c} + \text{ſin.}\frac{a}{c}\,\text{coſ.}\frac{x}{c}}{2\,\text{ſin.}\frac{a}{c}}.$$

88. Quia autem recta Cc eſt curvæ diameter, ponatur abſciſſa a puncto medio C ſumpta $CP = z$, erit $x = \frac{1}{2}a - z$. Unde

Unde fit $e^{\frac{x}{c}} = e^{\frac{a}{2c}} e^{-\frac{z}{c}} = e^{-\frac{z}{c}} \sqrt{\frac{1-\text{ſin.}\frac{a}{c}}{\text{coſ.}\frac{a}{c}}}$, &

$e^{-\frac{x}{c}} = e^{\frac{z}{c}} \sqrt{\frac{\text{coſ.}\frac{a}{c}}{1-\text{ſin.}\frac{a}{c}}}$; ex quo erit $\frac{Ae^{\frac{x}{c}}+Be^{-\frac{x}{c}}}{b} =$

$$\frac{(e^{\frac{z}{c}}+e^{-\frac{z}{c}})\sqrt{\text{coſ.}\frac{a}{c}(1-\text{ſin.}\frac{a}{c})}}{2(1-\text{ſin.}\frac{a}{x}+\text{coſ.}\frac{a}{c})} = \frac{e^{\frac{z}{c}}+e^{-\frac{z}{c}}}{2(e^{\frac{a}{2c}}+e^{-\frac{a}{2c}})}.$$

Tum vero erit $(1-\text{coſ.}\frac{a}{c})\text{ſin.}\frac{x}{c}+\text{ſin.}\frac{a}{c}\text{coſ.}\frac{x}{c} = \text{ſin.}\frac{x}{c}+\text{ſin.}\frac{a-x}{c} = \text{ſin.}(\frac{a}{2c}-\frac{z}{c})+\text{ſin.}(\frac{a}{2c}+\frac{z}{c}) = 2\,\text{ſin.}\frac{a}{2c}\,\text{coſ.}\frac{z}{c}$; quibus ſubſtitutis, oritur hæc æquatio:

$\frac{2y}{b} = \frac{e^{\frac{z}{c}}+e^{-\frac{z}{c}}}{e^{\frac{a}{2c}}+e^{-\frac{a}{2c}}} + \frac{\text{coſ.}\frac{z}{c}}{\text{coſ.}\frac{a}{2c}}$; quæ eſt forma ſimpliciſſima,

qua natura curvæ aMcb exprimi poteſt: manifeſtum autem eſt, ſive z ſumatur affirmative, ſive negative, eundem eſſe proditurum valorem applicatæ y. Eſt vero $e^{\frac{a}{2c}}+e^{-\frac{a}{2c}} = \frac{2\,\text{coſ.}\frac{a}{2c}}{\sqrt{\text{coſ.}\frac{a}{c}}}$. Invenimus autem angulum $\frac{a}{c} = 271^\circ, 0', 39\frac{1}{2}''$.

89. Si jam ponatur $z = 0$; præbebit y valorem applicatæ Cc; erit ergo $\frac{2.Cc}{b} = \frac{2\sqrt{\text{coſ.}\frac{a}{c}}}{2\,\text{coſ.}\frac{a}{2c}} + \frac{1}{\text{coſ.}\frac{a}{2c}}$ ſeu $\frac{Cc}{Aa}$

=

$$= \frac{1 + \sqrt{\text{coſ.}\,\frac{a}{c}}}{2\ \text{coſ.}\,\frac{a}{2c}} = \tfrac{1}{2}\,\text{ſec.}\,\frac{a}{2c} + \tfrac{1}{2}\,\text{ſec.}\,\frac{a}{2c}\sqrt{\text{coſ.}\,\frac{a}{c}}.$$ At eſt

$\text{coſ.}\,\frac{a}{c} = \text{ſin.}\ 1^\circ, 0', 39\frac{1}{4}''$ & $\text{coſ.}\,\frac{a}{2c} = \text{ſin.}\ 45^\circ, 30', 19\frac{7}{8}''$. Hinc reperitur $\frac{Cc}{Aa} = 0, 607815$. Deinde ſi ponatur $y = 0$, reperientur puncta E & F quibus curva axem interſecat. Erit ergo

$$e^{\frac{z}{c}} + e^{-\frac{z}{c}} = -\frac{\text{coſ.}\,\frac{z}{c}}{\text{coſ.}\,\frac{a}{2c}}\left(e^{\frac{a}{c}} + e^{-\frac{a}{2c}}\right) = \frac{2\,\text{coſ.}\,\frac{z}{c}}{\sqrt{\text{coſ.}\,\frac{a}{c}}},$$

ex qua per approximationes reperitur

$$\frac{CE}{CA} = 0, 551685, \ \& \ \frac{AE}{AC} = 0, 448315.$$

Dum ergo lamina oſcillationes peragit, hæc puncta E & F reſtabunt immobilia; ex quo hujuſmodi motus oſcillatorius, qui alias vix actu produci poſſe videatur, facile produci poterit. Si enim lamina in punctis E & F hoc modo definitis figatur, tum perinde oſcillabitur ac ſi penitus eſſet libera.

90. Si eodem modo tractetur æquationum ſupra inventarum ſecunda $\frac{a}{c} = \frac{7\pi}{2} + \Phi = l\ \text{cot.}\ \Phi$; quo quidem caſu reperietur proxime $\Phi = 0$; tum prodibit ſecundus modus, quo lamina libera vibrationes abſolvere poteſt, ſecando ſcilicet axem AB in quatuor punctis; ideoque lamina perinde oſcillabitur, ac ſi in his quatuor punctis eſſet fixa. Viciſſim ergo, ſi lamina in his quatuor punctis, vel eorum duobus tantum quibuſvis figatur; tum, eodem modo oſcillabitur ac ſi eſſet libera; ſonum autem edet multo auctiorem; quippe qui ad ſonum præcedentem modo editum rationem tenebit fere ut 7^2 ad 3^2, hoc eſt, intervallum erit duarum octavarum cum quarta & hemitonii ſemiſſe.

miſſe. Tertius oſcillandi modus, quo eſt $-\frac{a}{c} = \frac{11}{2}\pi + \Phi = l\ \text{cot.}\ \Phi$, habebit ſex curvæ acb interſectiones cum axe AB; ſonuſque edetur plus una octava cum tertia minore acutior; huncque ſonum lamina edet ſi in duobus illorum ſex punctorum figatur. Hinc patet quam varii ſoni ab eadem lamina, prout in duobus punctis diverſimode figitur, edi queant; & niſi puncta bina, quibus infigitur, congruant cum interſectionibus in modo primo, vel ſecundo, vel tertio, atque adeo oſcillationes ſeſe ad modorum aliquem ſequentium, vel etiam ad infiniteſimum componant, tum ſonum fore tantopere acutum, ut percipi omnino nequeat, ſeu quod eodem redit, lamina motum oſcillatorium prorſus recipere non poterit: vel ſaltem, inſtar cordæ vibrantis, cui ponticulus ita ſubjicitur ut partes nullam inter ſe teneant rationem rationalem, ſonus minus diſtinctus producetur.

Fig. 24. De oſcillationibus laminæ elaſticæ utroque termino fixæ.

90. Infixa nunc ſit lamina elaſtica in utroque termino A & B; ita tamen ut tangentes curvæ in his punctis non determinentur. Ad hunc ſcilicet caſum in experimentis producendum, laminæ in utroque termino infigantur tenuiſſimi aculei $A\alpha$, $B\beta$, qui parieti infixi reddant laminæ extremos terminos A & B immobiles. Ad motum oſcillatorium hujus laminæ elaſticæ inveſtigandum, ponatur, ut ante, elaſticitas abſoluta laminæ $= Ekk$, longitudo $AB = a$, & pondus $= M$, atque longitudo penduli ſimplicis iſochroni $= f$. Sit AMB figura curvilinea, quam lamina inter oſcillandum induit, ac ponatur abſciſſa $AP = AM = x$, applicata $PM = y$, & radius oſculi in $M = R$. Sit porro P vis, quam aculeus $A\alpha$ ſuſtinet in directione $A\alpha$, & quia vis, qua elementum Mm in directione $M\mu$ urgeri debet quo lamina in hoc ſtatu conſervetur, eſt $= \frac{Mydx}{af}$; erit, per Regulas ſupra deſcriptas, æquatio pro curva hæc

$$\frac{Ekk}{R} = Px - \frac{M}{af}\int dx \int y\, dx$$

Eſt vero $R = -\frac{dx^2}{ddy}$; quia curva verſus axem eſt concava;

Digitized by Google

unde fit $\frac{Ekkddy}{dx^2} = \frac{M}{af}\int dx\int y dx - Px$.

Facto ergo $x = 0$, erit radius oſculi R in A infinitus, ideoque $ddy = 0$.

91. Si hæc æquatio bis differentietur; prodibit eadem æquatio, quam pro caſibus præcedentibus invenimus,

$$Ekkd^4y = \frac{M}{af}ydx^4$$

Quod ſi ergo ponatur $\frac{Ekk\,af}{M} = c^4$, erit æquatio integralis

$$y = Ae^{\frac{x}{c}} + Be^{-\frac{x}{c}} + C\,\text{ſin.}\,\frac{x}{c} + D\,\text{coſ.}\,\frac{x}{c}.$$

Ad quam determinandam, ponatur $x = 0$, & quia ſimul y evaneſcere debet, erit $0 = A + B + D$.

Secundo, ponatur $x = a$, & quia pariter fieri debet $y = 0$, erit

$$0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} + C\,\text{ſin.}\,\frac{a}{c} + D\,\text{coſ.}\,\frac{a}{c}.$$

Tertio, quia $\frac{ddy}{dx^2}$ evaneſcere debet, poſito & $x = 0$ & $x = a$; fiet

$$0 = A + B - D, \;\&\; 0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} - C\,\text{ſin.}\,\frac{a}{c} - D\,\text{coſ.}\,\frac{a}{c}.$$

Jam æquationes $0 = A + B - D$ & $0 = A + B + D$ dant $D = 0$, & $B = -A$; qui valores in reliquis duabus æquationibus ſubſtituti præbent

$$0 = A(e^{\frac{a}{c}} - e^{-\frac{a}{c}}) + C\,\text{ſin.}\,\frac{a}{c}, \;\&\; 0 = A(e^{\frac{a}{c}} - e^{-\frac{a}{c}}) - C\,\text{ſin.}\,\frac{a}{c};$$

quibus ſatisfieri nequit, niſi ſit $A = 0$, quia non poteſt eſſe $e^{\frac{a}{c}} = e^{-\frac{a}{c}}$, præter caſum $\frac{a}{c} = 0$; tum vero eſſe debet $C\,\text{ſin.}\,\frac{a}{c} = 0$. Hic cum nequeat poni $C = 0$, quia motus oſcillatorius foret nullus, erit ſin. $\frac{a}{c} = 0$, ideoque vel

$\frac{a}{c}$

$\frac{a}{c} = \pi$, vel $\frac{a}{c} = 2\pi$, &c. unde iterum infiniti diverſi oſcillationum oriuntur modi, prout curva AMB axem vel nuſquam præter terminos A & B ſecat, vel in uno, vel in duobus, vel in pluribus punctis; uti colligitur ex æquatione $y = C \text{ ſin. } \frac{x}{c}$. Puncta interſectionum autem quotcunque fuerint, æqualibus intervallis inter ſe diſtabunt.

93. Cum igitur pro primo ac principali oſcillandi modo ſit $\frac{a}{c} = \pi$, erit $a^4 = \pi^4 c^4 = \pi^4 \times Ekk \times \frac{a}{M} \times f$, unde fit $f = \frac{a^4}{\pi^4} \times \frac{1}{Ekk} \times \frac{M}{a}$; Quare ratione longitudinis laminæ, ſoni iterum tenebunt rationem reciprocam duplicatam longitudinum. Sonus autem hujus laminæ hoc modo editus ſe habebit ad ſonum ejuſdem laminæ, ſi altero termino B muro eſſet infixa, uti $\pi\pi$ ad quadratum numeri 1, 8751040813, hoc eſt ut 2, 807041 ad 1, ſeu in numeris minimis ut 57 ad 160, quod intervallum eſt octava cum tritono fere. Si oſcillationes ſe ad ſecundum modum, quo eſt $\frac{a}{c} = 2\pi$, componant; ſonus fiet duplici octava acutior; ſin ſit $\frac{a}{c} = 3\pi$, ſonus acutior fiet tribus octavis cum tono majore, quam caſu quo $\frac{a}{c} = \pi$; & ita porro. Quæ quo facilius ad experimenta revocari queant; notandum eſt oſcillationes hic quam-minimas poni, ita ut nulla laminæ elongatione ſit opus. Quare, ne tenacitas laminæ, qua etiam minimæ extenſioni, ſine qua oſcillationes iſtæ peragi nequeunt, reluctatur, hic alterationem afferat; cuſpides illæ ita debent conſtitui, ut tantilla extenſio non impediatur: quod evenit ſi plano politiſſimo incumbant. Sic lamina elaſtica AB in A & B cuſpidibus A a & B C munita, ſi cuſpides ſpeculo imponantur, ſonum calculo conformem edet.

Digitized by Google

De oſcillationibus laminæ elaſticæ utroque termino parieti infixæ. *Fig.* 25.

94. Hoc caſu expedito; iſtam de laminis elaſticis tractationem claudat motus oſcillatorius laminæ elaſticæ, utroque termino A & B muro infixæ, ita ut inter oſcillandum puncta A & B non ſolum maneant immota, ſed etiam recta AB perpetuo ſit tangens curvæ AMB in punctis A & B. Hic ergo iterum cavendum eſt, ut obices terminos A & B comprehendentes non ſint adeo firmi, ſed tantillam extenſionem quanta ad curvaturam requiritur, permittant. Quæcunque ergo ſint vires in terminis A & B ad laminam continendam requiſitæ; ad ſequentem pervenietur æquationem differentialem quarti ordinis $Ekkd^4y = \frac{M}{af}\, y\, dx^4$; cujus, ſi ponatur $\frac{Ekk.af}{M} = c^4$, integralis erit ut ſupra,

$$y = Ae^{\frac{x}{c}} + Be^{-\frac{x}{c}} + C\,\text{ſin.}\,\frac{x}{c} + D\,\text{coſ.}\,\frac{x}{c}.$$

95. Conſtantes A, B, C, & D autem ita debent eſſe comparatæ, ut poſito $x = 0$, non ſolum y evaneſcat, ſed etiam fiat $dy = 0$, quia in A curva ab axe AB tangitur. Hoc idem utrumque vero evenire debet, ſi ponatur $x = a$; unde iſtæ quatuor æquationes naſcentur.

$$\text{I.}\quad 0 = A + B + D$$
$$\text{II.}\quad 0 = A - B + C$$
$$\text{III.}\quad 0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} + C\,\text{ſin.}\,\frac{a}{c} + D\,\text{coſ.}\,\frac{a}{c}$$
$$\text{IV.}\quad 0 = Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} + C\,\text{coſ.}\,\frac{a}{c} - D\,\text{ſin.}\,\frac{a}{c}.$$

Ex harum æquationum prima & ſecunda oritur $C = -A + B$, & $D = -A - B$, qui valores in reliquis duabus ſubſtituti dabunt

$$0 = Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} - (A - B)\,\text{ſin.}\,\frac{a}{c} - (A + B)\,\text{coſ.}\,\frac{a}{c}$$
$$0 = Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} - (A - B)\,\text{coſ.}\,\frac{a}{c} + (A + B)\,\text{ſin.}\,\frac{a}{c}$$

quarum

Digitized by Google

quarum ſumma ac differentia eſt,

$$o = Ae^{\frac{a}{c}} + B\,\text{ſin.}\,\frac{a}{c} - A\,\text{coſ.}\,\frac{a}{c},\ \text{ſeu}\ \frac{A}{B} = \frac{\text{ſin.}\,\frac{a}{c}}{\text{coſ.}\,\frac{a}{c} - e^{\frac{a}{c}}}$$

$$o = Be^{-\frac{a}{c}} - A\,\text{ſin.}\,\frac{a}{c} - B\,\text{coſ.}\,\frac{a}{c},\ \text{ſeu}\ \frac{A}{B} = \frac{e^{-\frac{a}{c}} - \text{coſ.}\,\frac{a}{c}}{\text{ſin.}\,\frac{a}{c}}$$

unde fit $2 = (e^{\frac{a}{c}} + e^{-\frac{a}{c}})\,\text{coſ.}\,\frac{a}{c}$, ſeu

$$e^{\frac{a}{c}} = \frac{1 \pm \text{ſin.}\,\frac{a}{c}}{\text{coſ.}\,\frac{a}{c}}.$$

Quæ æquatio, quia congruit cum ea, quam §. 81 invenimus, ſequentes Solutiones numero infinitæ ſatisfacient:

I. $\frac{a}{c} = \frac{1}{2}\pi - \Phi = l\,\text{cot.}\,\frac{1}{2}\Phi$

II. $\frac{a}{c} = \frac{3}{2}\pi + \Phi = l\,\text{cot.}\,\frac{1}{2}\Phi$

III. $\frac{a}{c} = \frac{5}{2}\pi - \Phi = l\,\text{cot.}\,\frac{1}{2}\Phi$

&c.

96. Harum æquationum primæ ſatisfieri nequit, niſi ſit $\Phi = 90°$, ideoque $\frac{a}{c} = 0$; unde primus oſcillandi modus oritur ex æquatione $\frac{a}{c} = \frac{3}{2}\pi + \Phi = l\,\text{cot.}\,\frac{1}{2}\Phi$; quæ cum jam ſupra ſit tractata, erit $\frac{a}{c} = 4,7300350232$. Quamobrem lamina elaſtica, cujus uterque terminus parieti infixus tenetur, perinde vibrationes ſuas peraget, ac ſi eſſet omnino libera. Hæc

autem convenientia tantum ad primum oſcillandi modum ſpectat; ſecundus enim oſcillandi modus, quo eſt $\frac{s}{c} = \frac{1}{2}\pi - \Phi =$ log. cot. $\frac{1}{2}\Phi$, atque lamina axem AB inter oſcillandum in uno puncto interſecat, in lamina libera ſui parem non habet; tertius autem modus laminæ utrinque infixæ congruet cum modo ſecundo laminæ liberæ, atque ita porro.

97. Hæc duo poſtrema oſcillationum genera, ob cauſam allatam, non congrue per experimenta explorari poſſunt: primum autem non ſolum ad experimenta inſtituenda maxime eſt aptum; ſed etiam adhiberi poteſt ad elaſticitatem abſolutam cujuſque laminæ propoſitæ, quam per Ekk indicavimus, inveſtigandam. Quod ſi enim ſonus notetur, quem hujuſmodi lamina altero termino muro infixa edit, eique in corda conſonus efficiatur, ſimul numerus oſcillationum uno minuto ſecundarum editarum cognoſcetur. Qui ſi æqualis ponatur expreſſioni $\frac{nn}{aa}\sqrt{g.Ekk.\frac{a}{M}}$, ob numerum n cognitum, & quantitates g, a, & M per dimenſiones inventas, reperietur valor expreſſionis Ekk; ſicque elaſticitas abſoluta innoteſcit; quæ cum ea quam ſupra ex incurvatione reperire docuimus, comparari poteſt.

AD-

ADDITAMENTUM II.

De motu projectorum in medio non reſiſtente, per Methodum maximorum ac minimorum determinando.

1. QUoniam omnes naturæ effectus ſequuntur quandam maximi minimive legem ; dubium eſt nullum , quin in lineis curvis , quas corpora projecta , ſi a viribus quibuſcunque ſollicitentur , deſcribunt , quæpiam maximi minimive proprietas locum habeat. Quænam autem ſit iſta proprietas , ex principiis metaphyſicis a priori definire non tam facile videtur : cum autem has ipſa curvas , ope Methodi directæ , determinare liceat ; hinc , debita adhibita attentione , id ipſum , quod in iſtis curvis eſt maximum vel minimum , concludi poterit. Spectari autem potiſſimum debet effectus a viribus ſollicitantibus oriundus ; qui cum in motu corporis genito conſiſtat , veritati conſentaneum videtur hunc ipſum motum , ſeu potius aggregatum omnium motuum qui in corpore projecto inſunt , minimum eſſe debere. Quæ concluſio etſi non ſatis confirmata videatur , tamen , ſi eam cum veritate jam a priori nota conſentire oſtendero , tantum conſequetur pondus , ut omnia dubia quæ circa eam ſuboriri queant penitus evaneſcant. Quin-etiam cum ejus veritas fuerit evicta , facilius erit in intimas Naturæ leges atque cauſas finales inquirere ; hocque aſſertum firmiſſimis rationibus corroborare.

2. Sit maſſa corporis projecti $= M$, ejuſque, dum ſpatiolum $= ds$ emetitur, celeritas debita altitudini $= v$; erit quantitas motus corporis in hoc loco $= M \sqrt{v}$; quæ per ipſum ſpatiolum ds multiplicata , dabit $M ds \sqrt{v}$ motum corporis collectivum per ſpatiolum ds. Jam dico lineam a corpore deſcriptam

criptam

Digitized by Google

curva ita erit comparata, ut in ea ſit $\int ds\sqrt{(a+gx)}$ minimum. Ponatur $dy = p\,dx$, ut ſit $ds = dx\sqrt{(1+pp)}$, atque minimum eſſe debet $\int dx\sqrt{(a+gx)(1+pp)}$; quæ expreſſio cum forma generali $\int Z\,dx$ comparata dat $Z = \sqrt{(a+gx)(1+pp)}$; quare, cum poſitum ſit $dZ = M\,dx + N\,dy + P\,dp$, erit $N = 0$ & $P = \frac{p\sqrt{(a+gx)}}{\sqrt{(1+pp)}}$. Quia ergo valor differentialis eſt $N - \frac{dP}{dx}$; ob $N = 0$, fiet præſenti caſu $dP = 0$, & $P = \sqrt{C}$. Habebitur ergo $\sqrt{C} = \frac{p\sqrt{(a+gx)}}{\sqrt{(1+pp)}} = \frac{dy\sqrt{(a+gx)}}{ds}$; unde fit $C\,dx^2 + C\,dy^2 = dy^2(a+gx)$, & $dy = \frac{dx\sqrt{C}}{\sqrt{(a-C+gx)}}$; quæ integrata dat $y = \frac{2}{g}\sqrt{C(a-C+gx)}$.

5. Manifeſtum quidem eſt hanc æquationem eſſe pro Parabola. At ejus conſenſum cum veritate attentius conſideraſſe juvabit. Primum ergo patet tangentem hujus curvæ eſſe horizontalem, ſeu $dx = 0$; ubi eſt $a - C + gx = 0$. Cum igitur principium abſciſſarum A ab arbitrio noſtro pendeat, ſumatur id in hoc ipſo loco, fietque $C = a$; tum vero ipſe axis per hoc punctum curvæ ſummum tranſeat, ita ut, poſito $x = 0$, fiat ſimul $y = 0$. His conſideratis, æquatio pro curva erit hæc $y = 2\sqrt{\frac{ax}{g}}$; quam non ſolum patet eſſe pro Parabola; ſed etiam, cum celeritas in puncto A ſit $\sqrt{a}$, altitudo CA, ex qua corpus labendo ab eadem vi g ſollicitatum eam ipſam acquirit celeritatem, qua in puncto A movetur, erit $= \frac{a}{g}$; hoc eſt, quartæ parametri parti æquatur; prorſus uti ex doctrina motus projectorum per Methodum directam intelligitur.

6. Sollicitetur, ut ante, corpus ubique verticaliter deorſum, et ipſa vis ſollicitans non ſit conſtans, ſed pendeat utcunque ab altitudine CP. Scilicet poſita abſciſſa $CP = x$, ſit vis qua corpus in M deorſum nititur $= X$ functioni cuicunque ipſius x. Si ergo vocetur applicata $PM = y$, elementum arcus

Digitized by Google

$Mm = ds$, & $dy = pdx$; erit $dv = Xdx$, & $v = A + \int Xdx$; unde minimum eſſe debet hæc expreſſio $\int dx \surd (A + \int Xdx)(1+pp)$, ex qua pro curva deſcripta AM obtinebitur hæc æquatio . .

$$\surd C = \frac{p\surd(A+\int Xdx)}{\surd(1+pp)} \;\&\; p = \frac{\surd C}{\surd(A - C + \int Xdx)} = \frac{dy}{dx};$$

ſeu $y = \int \frac{dx\surd C}{\surd(A - C + \int Xdx)}$. Tangens ergo curvæ erit horizontalis ubi $\int X\,dx = C - A$. Hæc vero eadem æquatio trajectoriæ corporis per Methodum directam reperitur.

Fig. 27. 7. Sollicitetur nunc corpus in M a duabus viribus, altera horizontali $= Y$ ſecundum directionem MP, altera verticali $= X$ ſecundum directionem MQ. Sit autem X functio quæcunque rectæ verticalis $MQ = CP = x$, & Y functio quæcunque applicatæ $PM = y$. Poſitis ergo ut ante $dy = pdx$, erit $dv = -Xdx - Ydy$, fietque $v = A - \int Xdx - \int Ydy$; unde minimum eſſe debet hæc formula $\int dx \surd(1+pp)(A - \int Xdx - \int Ydy)$. Differentietur $\surd(1+pp)(A - \int Xdx - \int Ydy)$, atque prodibit . . . .

$$\frac{-Xdx\surd(1+pp)}{2\surd(A - \int Xdx - \int Ydy)} - \frac{Ydy\surd(1+pp)}{2\surd(A - \int Xdx - \int Ydy)} + \frac{pdp\surd(A - \int Xdx - \int Ydy)}{\surd(1+pp)}.$$ Hinc poſito

$$N = \frac{-Y\surd(1+pp)}{2\surd(A - \int Xdx - \int Ydy)}, \;\&\; P = \frac{p\surd(A - \int Xdx - \int Ydy)}{\surd(1+pp)};$$

erit pro curva quæſita hæc æquatio $0 = N - \frac{dP}{dx}$, ſeu $Ndx = dP$. Hinc ergo fit $\frac{-Ydx\surd(1+pp)}{2\surd(A - \int Xdx - \int Ydy)} =$

$$\frac{dp\surd(A - \int Xdx - \int Ydy)}{(1+pp)\surd(1+pp)} - \frac{pXdx - pYdy}{2\surd(1+pp)(A - \int Xdx - \int Ydy)}$$

ſeu $\frac{dp\surd(A - \int Xdx - \int Ydy)}{(1+pp)\surd(1+pp)} = \frac{Xdy - Ydx}{2\surd(1+pp)(A - \int Xdx - \int Ydy)}$;

ideoque $\frac{2dp}{1+pp} = \frac{Xdy - Ydx}{A - \int Xdx - \int Ydy}$. Hanc æquationem veritati eſſe conſentaneam patebit, ſi loco $A - \int Xdx - \int Ydy$

ponatur

ponatur v, erit enim $\frac{2\,v\,dp}{(1+pp)^{3:2}} = \frac{X\,dy - Y\,dx}{\sqrt{(1+pp)}}$. At eſt radius oſculi $r = -\frac{(1+pp)^{3:2}\,dx}{dp}$, quo introducto eſt $\frac{2v}{r} = \frac{Y\,dx - X\,dy}{ds}$; ubi eſt $\frac{2v}{r}$ vis corporis centrifuga, & $\frac{Y\,dx - X\,dy}{ds}$ exprimit vim normalem ex viribus ſollicitantibus ortam; quarum virium æqualitas utique in omni motu projectorum locum habet.

8. Æquatio autem inventa $\frac{2\,dp}{1+pp} = \frac{X\,dy - Y\,dx}{A - \int X\,dx - \int Y\,dy}$ ita generaliter eſt integrabilis, ſi multiplicetur per $\frac{p(A - \int X\,dx - \int Y\,dy)}{1+pp}$; fiet enim $\frac{2p\,dp\,(A - \int X\,dx - \int Y\,dy)}{(1+pp)^2} - \frac{pp\,X\,dx + Y\,dy}{1+pp} = 0$; quæ integrata dat $\frac{-p^2\int X\,dx + \int Y\,dy - A}{1+pp} = C$, ſeu $\int Y\,dy - p^2\int X\,dx = A + C + Cpp$, unde $p = \frac{\sqrt{(B + \int Y\,dy)}}{\sqrt{(C + \int X\,dx)}}$; poſito B pro $-A-C$. Cum ergo ſit $p = \frac{dy}{dx}$, erit $\int \frac{dy}{\sqrt{(B + \int Y\,dy)}} = \int \frac{dx}{\sqrt{(C + \int X\,dx)}}$, æquatio pro curva quæſita, in qua variabiles x & y ſunt a ſe invicem ſeparatæ. Vel ſi conſtantes B & C in negativas convertantur, erit $\int \frac{dy}{\sqrt{(B - \int Y\,dy)}} = \int \frac{dx}{\sqrt{(A - \int X\,dx)}}$. Ex quibus etſi curvæ conſtructio facilis habetur, tamen æquationes algebraïcæ, quoties quidem in ipſis continentur, non tam facile eruuntur. Sint X & Y functiones ſimiles & quidem poteſtates ipſarum x & y, ita ut ſit $\int \frac{dy}{\sqrt{(b^n - y^n)}} = \int \frac{dx}{\sqrt{(a^n - x^n)}}$, quæ æquatio, ſi $n = 1$, præbet Parabolam; ſin $n = 2$, Ellipſin centrum in C habentem: etſi hoc caſu utraque integratio quadraturam Circuli

Digitized by Google

requirit. Veriſimile ergo videtur etiam aliis caſibus, quibus neutra integratio ſuccedit, curvas algebraïcas ſatisfacere; quarum autem inveniendarum Methodus adhuc deſideratur.

9. Urgeatur corpus M perpetuo verſus punctum fixum ſecundum directionem MC, vi quæ ſit ut functio quæcunque diſtantiæ MC. Poſitis ut ante $CP = x$, $PM = y$, & $dy = p\,dx$; ſit $CM = \sqrt{(x^2 + y^2)} = t$, atque ſit T ea functio ipſius t, quæ exprimit vim centripetam. Reſolvatur hæc vis in laterales ſecundum MQ & MP, erit vis trahens ſecundum $MQ = \frac{Tx}{t}$; & vis ſecundum $MP = \frac{Ty}{t}$; ex quibus oritur acceleratio $dv = -\frac{Tx\,dx}{t} - \frac{Ty\,dy}{t} = -T\,dt$, ob $x\,dx + y\,dy = t\,dt$; unde fit $v = A - \int T\,dt$. Quamobrem minimum eſſe debet hæc expreſſio $\int dx\sqrt{(1+pp)}\,(A - \int T\,dt)$. Jam, ſecundum Regulæ præceptum, differentietur quantitas, $\sqrt{(1+pp)}\,(A - \int T\,dt)$, prodibitque

$$-\frac{T\,dt\sqrt{(1+pp)}}{2\sqrt{(A - \int T\,dt)}} + \frac{p\,dp\sqrt{(A - \int T\,dt)}}{\sqrt{(1+pp)}}.$$

Ob $dt = \frac{x\,dx + y\,dy}{t}$, erit ergo $N = \frac{-Ty\sqrt{1+pp}}{2t\sqrt{(A - \int T\,dt)}}$ & $P = \frac{p\sqrt{(A - \int T\,dt)}}{\sqrt{(1+pp)}}$; ex quibus efficitur æquatio pro curva $N\,dx = dP$, quæ præbet,

$$-\frac{Ty\,dx\sqrt{(1+pp)}}{2t\sqrt{(A - \int T\,dt)}} = \frac{dp\sqrt{(A - \int T\,dt)}}{(1+pp)\sqrt{(1+pp)}} - \frac{p\,T\,dt}{2\sqrt{(1+pp)}(A - \int T\,dt)}$$

hæcque reducta abibit in iſtam,

$$\frac{T(x\,dy - y\,dx)}{2t(A - \int T\,dt)} = \frac{dp}{1+pp}.$$

10. Quamvis hæc æquatio quatuor contineat litteras diverſas, tamen debita dexteritate integrari poteſt. Cum enim ſit $y\,dy + x\,dx = t\,dt = py\,dx + x\,dx$, erit $dx = \frac{t\,dt}{x+py}$ & $dy = \frac{pt\,dt}{x+py}$, qui valores in æquatione ſubſtituti dabunt

$$\frac{-(px - y)\,T\,dt}{2(x+py)(A - \int T\,dt)} = \frac{dp}{1+pp}, \text{ ſeu } \frac{T\,dt}{2(A - \int T\,dt)} = \frac{dp(py + x)}{(1+pp)(px - y)}.$$

Ha-

Digitized by Google

Harum expreſſionum utraque per logarithmos eſt integrabilis; eſt enim $\int \frac{Tdt}{2(A - \int Tdt)} = -\frac{1}{2}\, l(A - \int Tdt)$, & $\int \frac{dp(x + py)}{(1+pp)(px - y)}$ reſolvitur in $\int \frac{xdp}{px - y} - \int \frac{pdp}{1+pp} = l\, \frac{px - y}{\surd(1+pp)}$; ita ut ſit $\frac{C}{\surd(A - \int Tdt)} = \frac{px - y}{\surd(1+pp)}$; qua æquatione declaratur, celeritatem corporis in M, quæ eſt $= \surd(A - \int Tdt)$, eſſe reciproce ut perpendiculum ex C in tangentem demiſſum; quæ eſt proprietas palmaria horum motuum.

11. Hoc vero idem Problema commodius reſolvi poteſt ipſam rectam CM pro altera variabili aſſumendo. Verum Methodus ſupra tradita non poſtulat, ut ambæ variabiles ſint coordinatæ orthogonales, dummodo ſint ejuſmodi binæ quantitates quibus determinatis ſimul curvæ punctum determinetur. Hanc ob cauſam, non liceret diſtantiam CM cum perpendiculo ex C in tangentem demiſſo pro binis illis variabilibus accipere; quoniam etiamſi detur & diſtantia a centro & perpendiculum in tangentem, hinc tamen locus puncti curvæ non definitur. Nihil autem impedit, quo-minus diſtantia CM, & arcus circuli BP centro C deſcripti, in locum duarum variabilium ſubſtituantur; quia dato arcu BP, & diſtantia CM curvæ punctum M æque determinatur ac per coordinatas orthogonales. Hac ergo annotatione uſus Methodi multo latius extenditur, quam alioquin videri queat. *Fig.* 28.

12. Sit igitur diſtantia corporis a centro $MC = x$, & vis qua corpus ad centrum C ſollicitatur ſit $= X$ functioni cuicunque ipſius x. Centro C, radio pro lubitu aſſumpto $BC = c$, deſcribatur circulus, cujus arcus BP teneat locum alterius variabilis y, ita ut ſit $Pp = dy = pdx$. Ex vi autem ſollicitante eſt $dv = -Xdx$, unde $v = A - \int Xdx$. Centro C, radio $CM = x$, deſcribatur arculus Mn, erit $mn = dx$; & CP : Pp = CM : Mn, unde fit $Mn = \frac{pxdx}{c}$, & elementum ſpatii $Mm = dx\surd(1 + \frac{ppxx}{cc})$. Quamobrem minimum eſſe

debet hæc formula $\int dx \sqrt{(A - \int X dx)(1 + \frac{ppxx}{cc})}$; ex qua oritur valor differentialis $\frac{1}{dx} d. \frac{pxx\sqrt{(A - \int X dx)}}{c\sqrt{(cc + ppxx)}}$, qui, per Regulam, nihilo æqualis poſitus, præbebit hanc æquationem:
$\sqrt{C} = \frac{pxx\sqrt{(A - \int X dx)}}{c\sqrt{(cc + ppxx)}}$, ſeu $Cc^4 + Cccppxx = (A - \int X dx)ppx^4$, ex qua fit

$$p = \frac{cc\sqrt{C}}{\sqrt{((A - \int X dx)x^4 - Ccxx)}} = \frac{cc\sqrt{C}}{x\sqrt{((A - \int X dx)xx - Ccc)}}$$

ſeu $dy = \frac{ccdx\sqrt{C}}{x\sqrt{((A - \int X dx)xx - Ccc)}}$, quæ eadem æquatio etiam per Methodum directam invenitur.

13. Ex his igitur caſibus perfectiſſimus conſenſus principii hic ſtabiliti cum veritate elucet: utrum autem iſte conſenſus in caſibus magis complicatis locum quoque ſit habiturus, dubium ſupereſſe poteſt. Quamobrem quam late pateat iſtud principium diligentius erit inveſtigandum, quo plus ipſi non tribuatur quam ejus natura permittit. Ad hoc explicandum, omnis motus projectorum in duo genera diſtribui debet; quorum altero celeritas corporis, quam in quavis loco habet, a ſolo ſitu pendet; ita ut, ſi ad eundem ſitum revertatur, eandem quoque ſit recuperaturum celeritatem; quod evenit, ſi corpus vel ad unum vel ad plura centra fixa trahatur viribus, quæ ſint ut functiones quæcunque diſtantiarum ab his centris. Ad alterum genus refero eos projectorum motus, quibus celeritas corporis per ſolum locum in quo hæret non determinatur; id quod uſu venit, vel ſi centra illa ad quæ corpus ſollicitatur fuerint mobilia, vel ſi motus fiat in medio reſiſtente. Hac facta diviſione; notandum eſt, quoties motus corporis ad prius genus pertineat, hoc eſt, ſi corpus non ſolum ad unum ſed ad quotcunque centra fixa ſollicitetur viribus quibuſcunque, toties in motu hoc ſummam omnium motuum elementarium fore minimam.

14. Hoc ipſum autem poſtulat indoles Propoſitionis: dum enim, inter datos terminos, ea quæritur curva, in qua ſit $\int ds\sqrt{v}$ minimum; eo ipſo aſſumitur, celeritatem corporis in utroque

termino

termino eandem eſſe, quæcunque curva corporis viam conſtituat. Quotcunque autem fuerint centra virium fixa, celeritas corporis in quovis loco M, exprimitur functione determinata ambarum variabilium $CP = x$, & $PM = y$. Sit igitur v functio quæcunque ipſarum x & y, ita ut ſit $dv = Tdx + Vdy$; atque videamus, an principium noſtrum veram exhibiturum ſit projectoriam corporis. Cum autem ſit $dv = Tdx + Vdy$; corpus perinde movebitur, ac ſi ſollicitetur in M a duabus viribus, altera T in directione abſciſſis x parallela, altera vero V in directione parallela applicatis y, ex quibus oritur vis tangentialis $= \frac{Tdx + Vdy}{ds}$, & vis normalis $= \frac{-Vdx + Tdy}{ds}$. Debet autem, ex natura motus liberi, eſſe $\frac{2v}{r} = \frac{-Vdx + Tdy}{ds} = \frac{-V + Tp}{\sqrt{(1+pp)}}$; ad quam æquationem ſi Methodus maximorum ac minimorum deducat, erit utique principium noſtrum veritati conforme. Fig. 27.

15. Cum igitur, per hoc principium, debeat eſſe $\int dx \sqrt{v(1+pp)}$ minimum, differentietur quantitas $\sqrt{v(1+pp)}$, atque, ob $dv = Tdx + Vdy$, orietur:

$$\frac{Tdx\sqrt{(1+pp)}}{2\sqrt{v}} + \frac{Vdy\sqrt{(c+pp)}}{2\sqrt{v}} + \frac{pdp\sqrt{v}}{\sqrt{(1+pp)}},$$

ex quo obtinetur pro curva quæſita ſequens æquatio, ſecundum præcepta tradita,

$$\frac{Vdx\sqrt{(1+pp)}}{2\sqrt{v}} = d.\frac{p\sqrt{v}}{\sqrt{(1+pp)}} = \frac{dp\sqrt{v}}{(1+pp)^{3:2}} + \frac{p(Tdx + Vdy)}{2\sqrt{v(1+pp)}}$$

ſeu $\frac{-dp\sqrt{v}}{(1+pp)^{3:2}} = \frac{Tpdx - Vdx}{2\sqrt{v(1+pp)}}$. At eſt radius oſculi in $M = \frac{-(1+pp)dx\sqrt{(1+pp)}}{dp}$; qui ſi ponatur $= r$, erit $\frac{2v}{r} = \frac{Tp - V}{\sqrt{(1+pp)}}$; omnino uti per Methodum directam invenitur. Dummodo ergo vires ſollicitantes ita fuerint comparatæ, ut eæ reduci queant ad duas vires T & V, ſecundum directiones coordinatis x & y parallelas ſollicitantes, quæ ſint ut functiones quæcunque harum variabilium x & y, tum ſemper in

curva

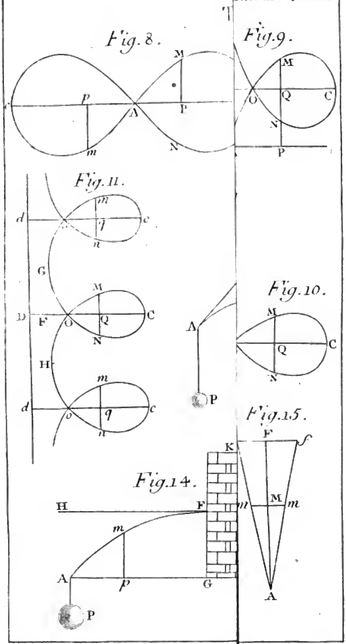
curva deſcripta erit motus corporis per omnia elementa collectus minimus.

16. Tam late ergo hoc principium patet, ut ſolus motus a reſiſtentia medii perturbatus excipiendus videatur; cujus quidem exceptionis ratio facile perſpicitur, propterea quod hoc caſu corpus per varias vias ad eundem locum perveniens non eandem acquirit celeritatem. Quamobrem, ſublata omni reſiſtentia in motu corporum projectorum, perpetuo hæc conſtans proprietas locum habebit, ut ſumma omnium motuum elementarium ſit minima. Neque vero hæc proprietas in motu unius corporis tantum cernetur, ſed etiam in motu plurium corporum conjunctim; quæ quomodocunque in ſe invicem agant, tamen ſemper ſumma omnium motuum eſt minima. Quod, cum hujuſmodi motus difficulter ad calculum revocentur, facilius ex primis principiis intelligitur, quam ex conſenſu calculi ſecundum utramque Methodum inſtituti. Quoniam enim corpora, ob inertiam, omni ſtatus mutationi reluctantur; viribus ſollicitantibus tam parum obtemperabunt, quam fieri poteſt, ſiquidem ſint libera; ex quo efficitur, ut, in motu genito, effectus a viribus ortus minor eſſe debeat, quam ſi ullo alio modo corpus vel corpora fuiſſent promota. Cujus ratiocinii vis, etiamſi nondum ſatis perſpiciatur; tamen, quia cum veritate congruit, non dubito quin, ope principiorum ſanioris Metaphyſicæ, ad majorem evidentiam evehi queat; quod negotium aliis, qui Metaphyſicam profitentur, relinquo.

INDEX

Digitized by Google

Additamentum.

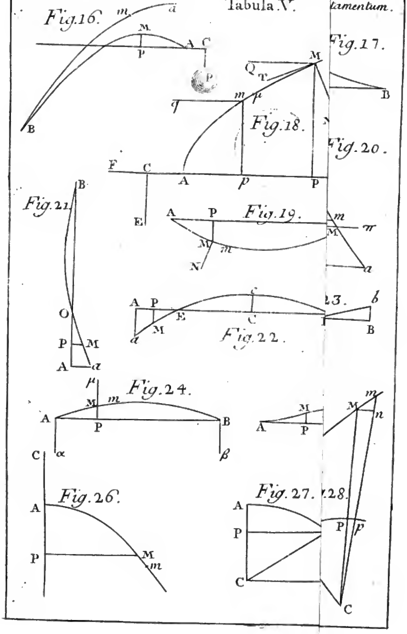


Digitized by Google

BIBLIOTECA NAZIONALE VITTORIO EMANUELE III
NAPOLI

Digitized by Google

Tabula V. [Supple]mentum.



Digitized by Google

BIBLIOTECA NAZ. "VITTORIO EMANUELE III" NAPOLI

Digitized by Google

INDEX

CAP. I. De Methodo maximorum & minimorum ad lineas curvas inveniendas applicata in genere, *pag.* 1

CAP. II. De Methodo maximorum & minimorum ad lineas curvas inveniendas abſoluta, 31

CAP. III. De inventione curvarum maximi minimive proprietate præditarum, ſi in ipſa maximi minimive formula inſunt quantitates indeterminatæ, 83

CAP. IV. De uſu Methodi hactenus traditæ in reſolutione varii generis quæſtionum, 129

CAP. V. Methodus, inter omnes curvas eadem proprietate præditas, inveniendi eam quæ maximi minimive proprietate gaudeat, 171

Digitized by Google

Cap. VI. Methodus, inter omnes curvas pluribus proprietatibus communibus gaudentes, eam determinandi quæ maximi minimive proprietate prædita ſit, 227

Additamentum I. De Curvis elaſticis, 245

Additamentum II. De Motu Projectorum in medio non reſiſtente, per Methodum maximorum ac minimorum determinando, 311

FINIS.

606829
SBN

Digitized by Google

Monitum ad Bibliopegam.

Tabulæ omnes Figurarum ad calcem compingantur, vel duæ priores poſt paginam 244 inſerantur, poſteriores tres ad calcem ponantur.

Avis au Relieur.

Il placera les cinq Planches de Figures à la fin du Livre, ou bien, il mettra les deux premieres à la page 244, & les trois dernieres à la fin.

Digitized by Google

Digitized by Google

Digitized by Google

Digitized by Google

Digitized by Google